

# Sorting Algorithms

Now that you've learned about arrays and linked lists, are you Team Array or Team Linked List?

[pollev.com/cs106bpoll](https://pollev.com/cs106bpoll)



# Team Array or Team Linked List... and why?

Top



# Roadmap

C++ basics

User/client

vectors + grids

stacks + queues

sets + maps

Core  
Tools

testing

algorithmic  
analysis

recursive  
problem-solving

Object-Oriented  
Programming

Implementation

arrays

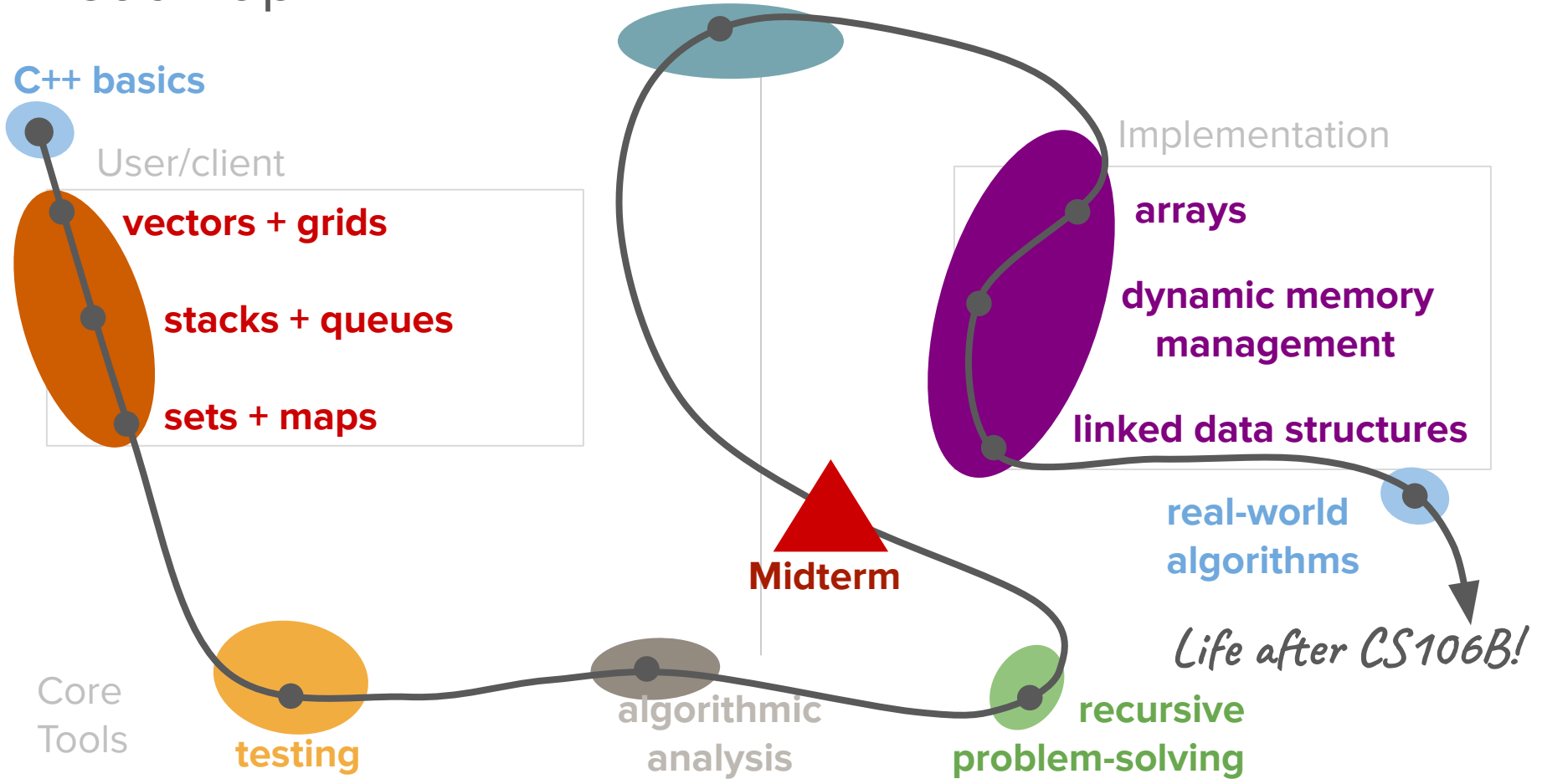
dynamic memory  
management

linked data structures

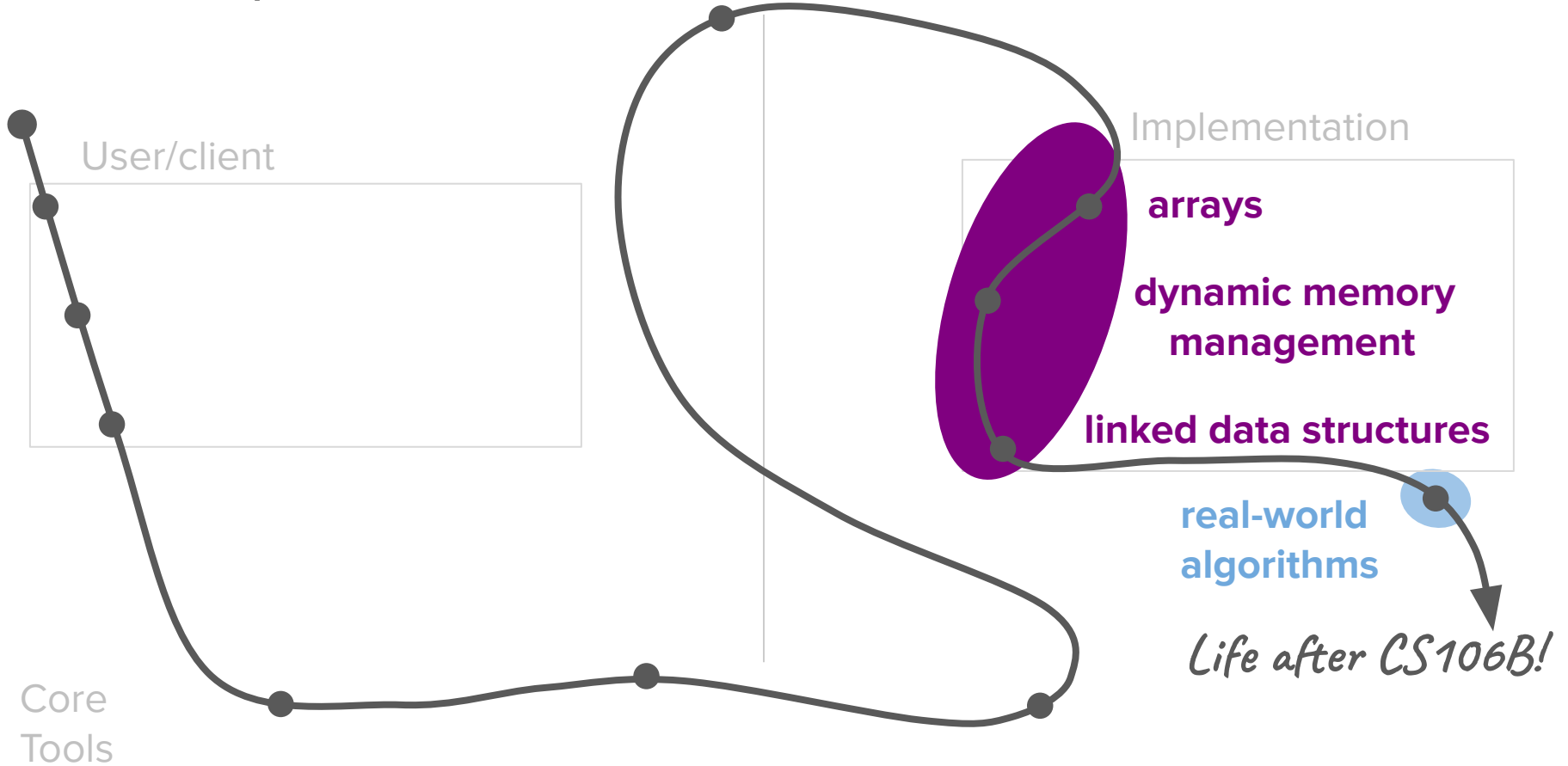
real-world  
algorithms

*Life after CS106B!*

Midterm



# Roadmap





# Today's questions

What are some real-world algorithms that can be used to organize data?

How can we design better, more efficient sorting algorithms?

# Today's topics

1. Review
2. Introduction to Sorting
3. Selection Sort
4. Divide-and-Conquer Sorts  
(MergeSort and QuickSort)

# Review

[linked list operations]



# Common linked lists operations

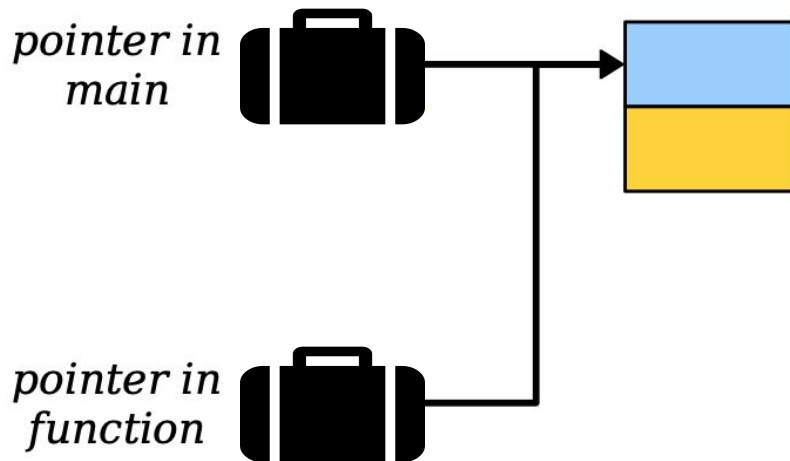
- **Traversal**
  - How do we walk through all elements in the linked list?
- **Rewiring**
  - How do we rearrange the elements in a linked list?
- **Insertion**
  - How do we add an element to a linked list?
- **Deletion**
  - How do we remove an element from a linked list?

# Linked List Traversal Takeaways

- Temporary pointers into lists are very helpful!
  - When processing linked lists iteratively, it's common to introduce pointers that point to cells in multiple spots in the list.
  - This is particularly useful if we're destroying or rewiring existing lists.
- Using a **while** loop with a condition that checks to see if the current pointer is **nullptr** is the prevailing way to traverse a linked list.
- Iterative traversal offers the most flexible, scalable way to write utility functions that are able to handle all different sizes of linked lists.

# Pointers by Value

- Unless specified otherwise, function arguments in C++ are passed by value – this includes pointers!
- A function that takes a pointer as an argument gets a copy of the pointer.
- We can change where the copy points, but not where the original pointer points.



# Pointers by Reference

- To solve this problem, we can **pass the linked list pointer by reference.**
- The mechanics of how to do so:

```
void prependTo(Node*& list, string data) {  
    Node* newNode = new Node;  
    newNode->data = data;  
  
    newNode->next = list;  
    list = newNode;  
}
```

*This is a **reference to a pointer to a Node**. If we change where list points in this function, the changes will stick!*

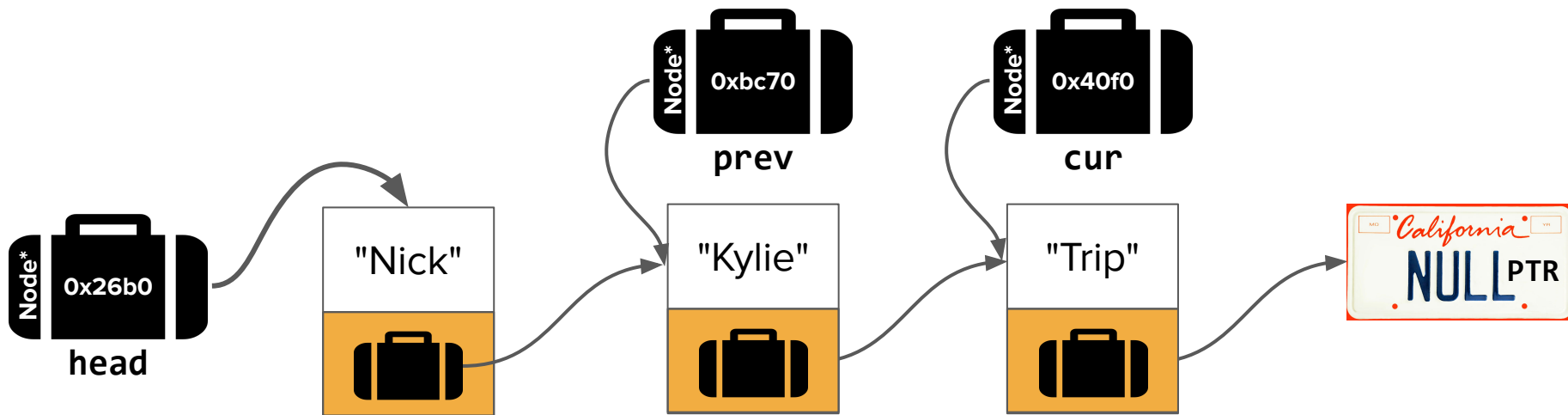
## A more efficient `appendTo()` - using a tail pointer!

```
Node* createListWithTailPtr(Vector<string> values) {
    if (values.isEmpty()) return nullptr;
    Node* head = new Node(values[0], nullptr);

    Node* cur = head;
    for (int i = 1; i < values.size(); i++) {
        Node* newNode = new Node(values[i], nullptr);
        cur->next = newNode;
        cur = newNode;
    }
    return head;
}
```

# Takeaways for manipulating the middle of a list

- While traversing to where you want to add/remove a node, you'll often want to keep track of both a current pointer and a previous pointer.
  - This makes rewiring easier between the two!
  - This also means you have to check that neither is nullptr before dereferencing.



# Linked list summary

- We saw lots of ways to examine and manipulate linked lists!
  - Traversal
  - Rewiring
  - Insertion (front/back/middle)
  - Deletion (front/back/middle)
- We saw linked lists in classes and outside classes, and pointers passed by value and passed by reference.
- Assignment 5 will really test your understanding of linked lists.
  - Draw lots of pictures!
  - Test small parts of your code at a time to make sure individual operations are working correctly.

# Sorting



What are some real-world algorithms that can be used to organize data?

What is sorting?

This is one kind of sorting...



This is one kind of sorting...but not quite what we mean!



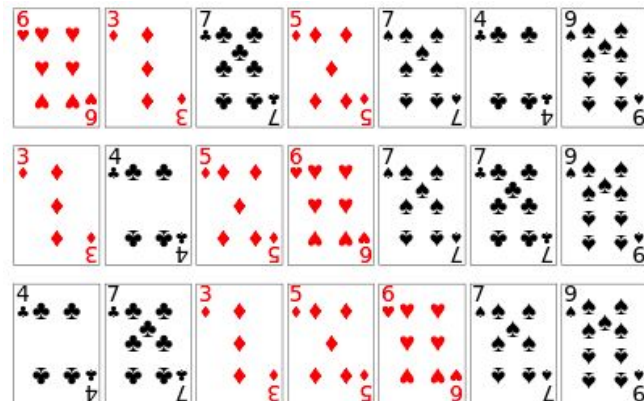
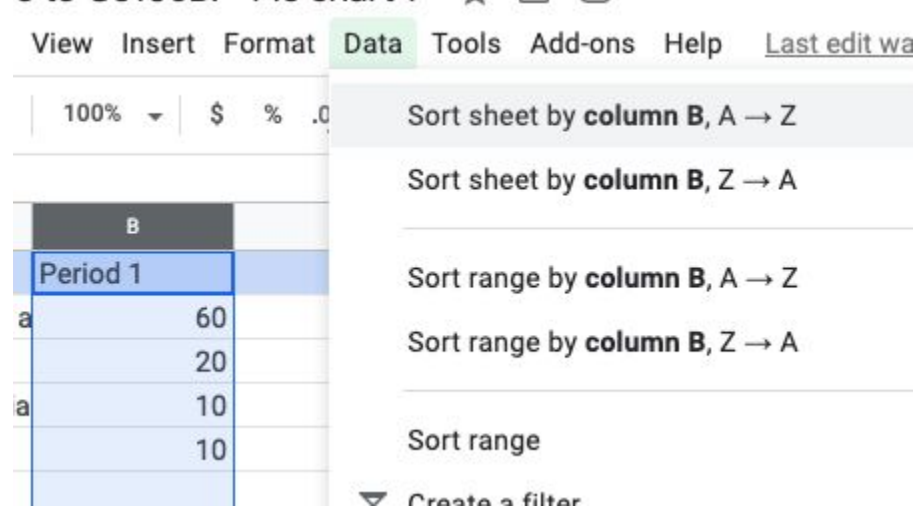
## *Definition*

### **sorting**

Given a list of data points, sort those data points into ascending / descending order by some quantity.














































# Why is sorting useful?

Time	Auto	Athlete	Nationality	Date	Venue
4:37.0		Anne Smith	United Kingdom	3 June 1967 <sup>[8]</sup>	London
4:36.8		Maria Gommers	Netherlands	14 June 1969 <sup>[8]</sup>	Leicester
4:35.3		Ellen Tittel	West Germany	20 August 1971 <sup>[8]</sup>	Sittard
4:29.5		Paola Pigni	Italy	8 August 1973 <sup>[8]</sup>	Viareggio
4:23.8		Natalia Mărășescu	Romania	21 May 1977 <sup>[8]</sup>	Bucharest
4:22.1	4:22.09	Natalia Mărășescu	Romania	27 January 1979 <sup>[8]</sup>	Auckland
4:21.7	4:21.68	Mary Decker	United States	26 January 1980 <sup>[8]</sup>	Auckland
	4:20.89	Lyudmila Veselkova	Soviet Union	12 September 1981 <sup>[8]</sup>	Bologna
	4:18.08	Mary Decker-Tabb	United States	9 July 1982 <sup>[8]</sup>	Paris
	4:17.44	Maricica Puică	Romania	9 September 1982 <sup>[8]</sup>	Rieti
	4:16.71	Mary Decker-Slaney	United States	21 August 1985 <sup>[8]</sup>	Zürich
	4:15.61	Paula Ivan	Romania	10 July 1989 <sup>[8]</sup>	Nice
	4:12.56	Svetlana Masterkova	Russia	14 August 1996 <sup>[8]</sup>	Zürich
	4:12.33	Sifan Hassan	Netherlands	12 July 2019	Monaco



# Approaches to sorting

- Suppose we want to rearrange a sequence to put elements into ascending order (each element is less than or equal to the element that follows it).
- In this lecture, we're going to answer the following questions:
  - What are some strategies we could use?
  - How do those strategies compare?
  - Is there a “best” strategy?

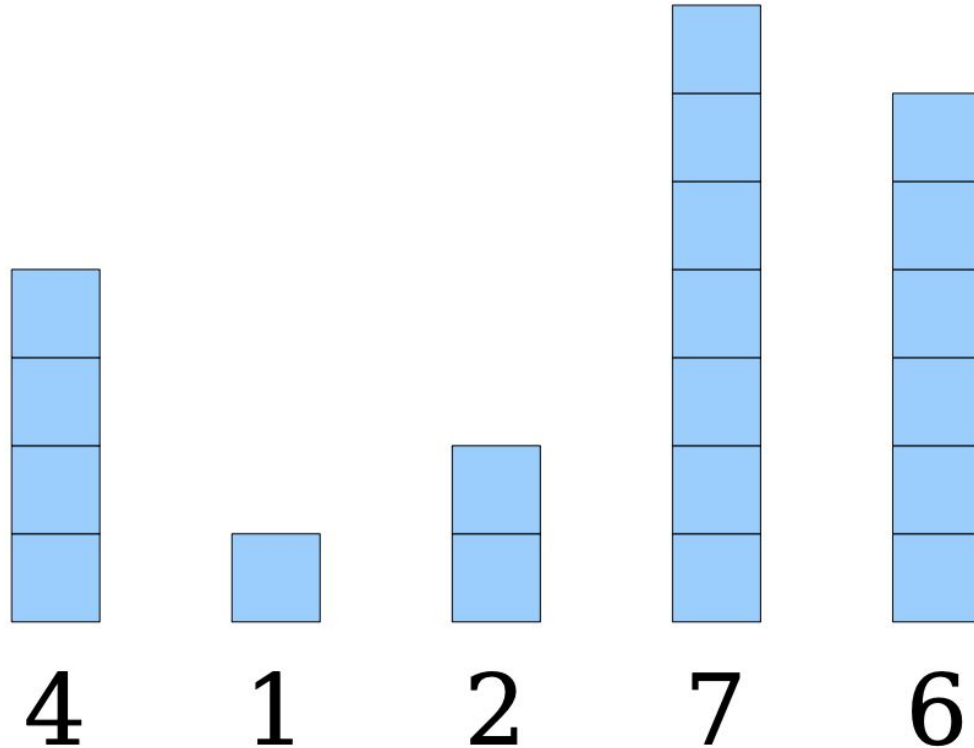
 <b>Play All</b>	 Insertion	 Selection	 Bubble	 Shell	 Merge	 Heap	 Quick	 Quick3
 Random								
 Nearly Sorted								
 Reversed								
 Few Unique								



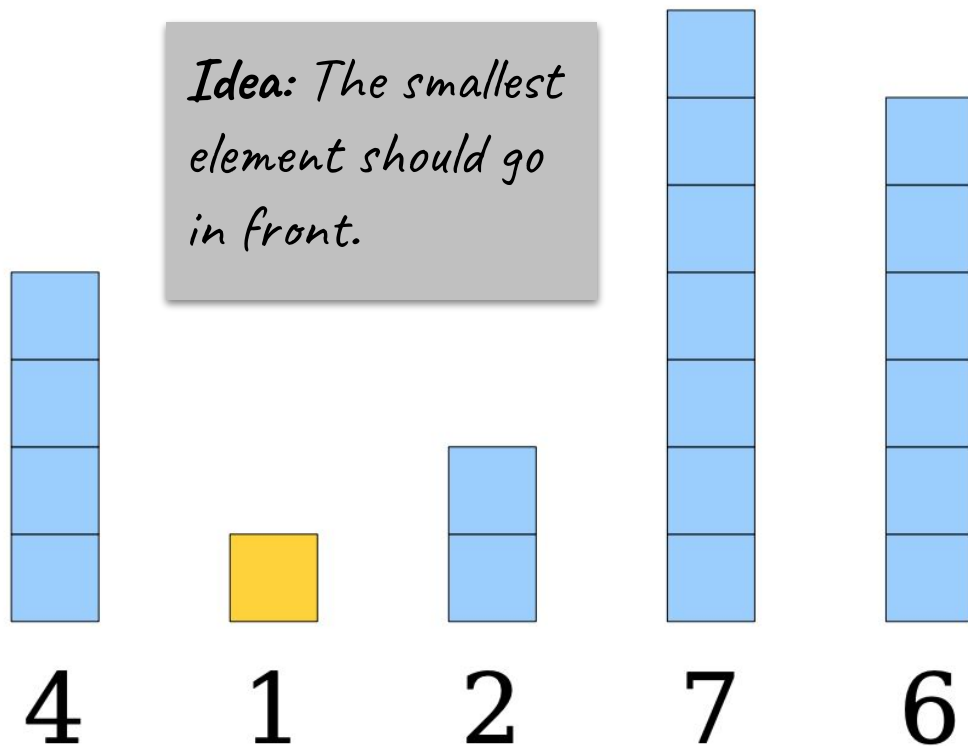
# Sorting algorithms

*Animations courtesy of Keith Schwarz!*

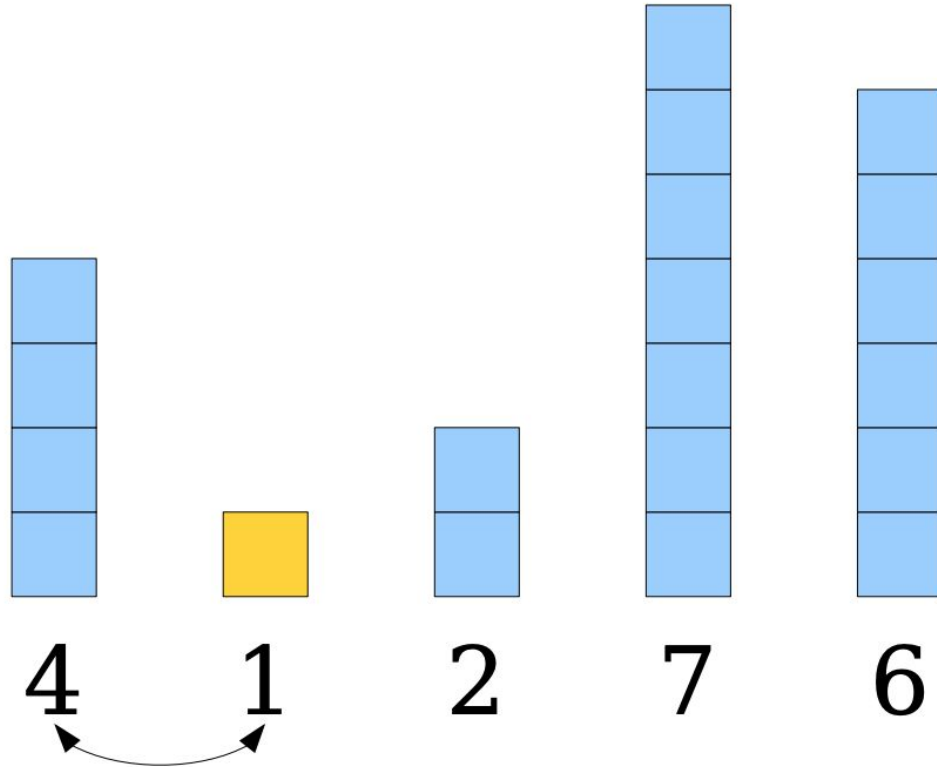
# Our first sort: Selection sort



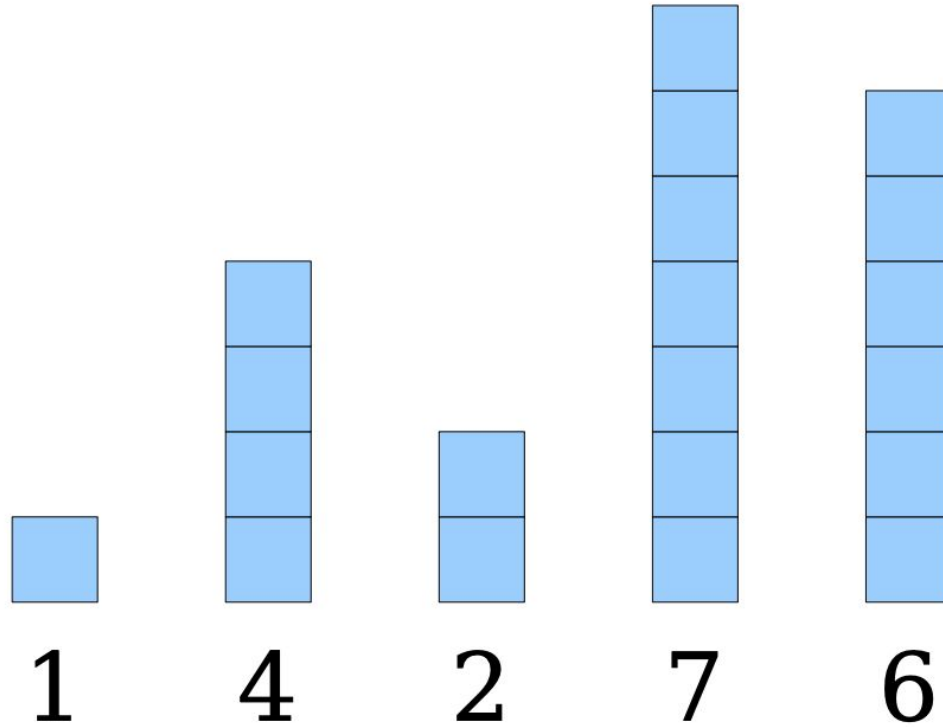
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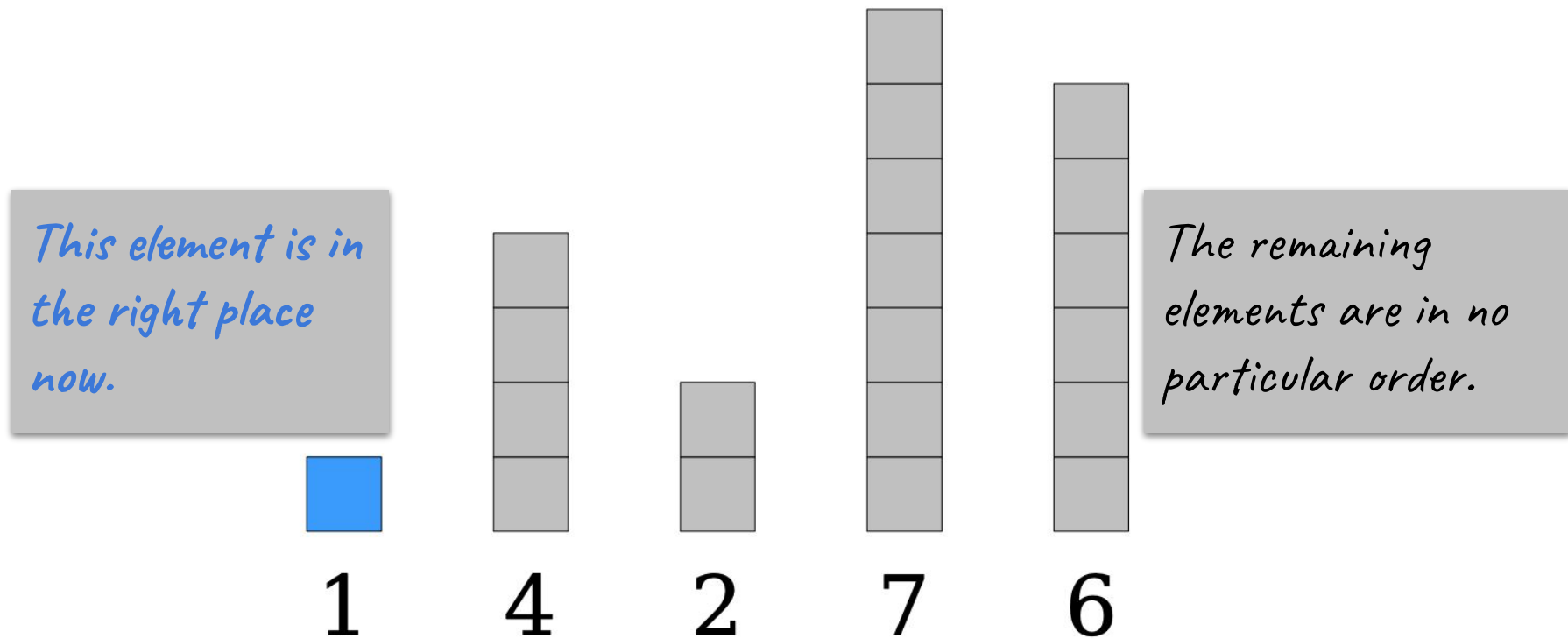
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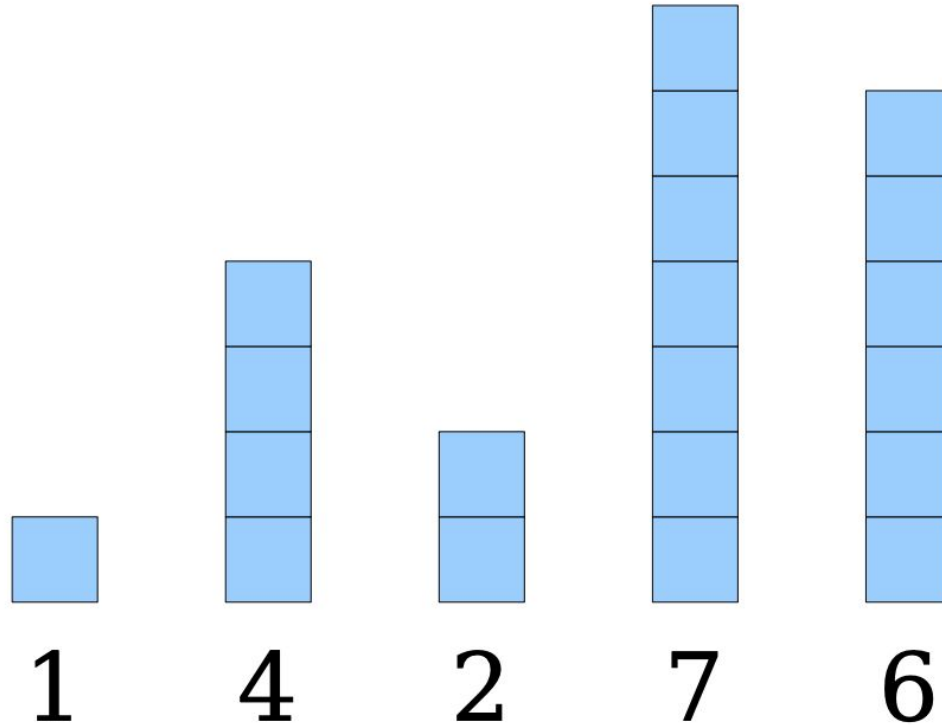
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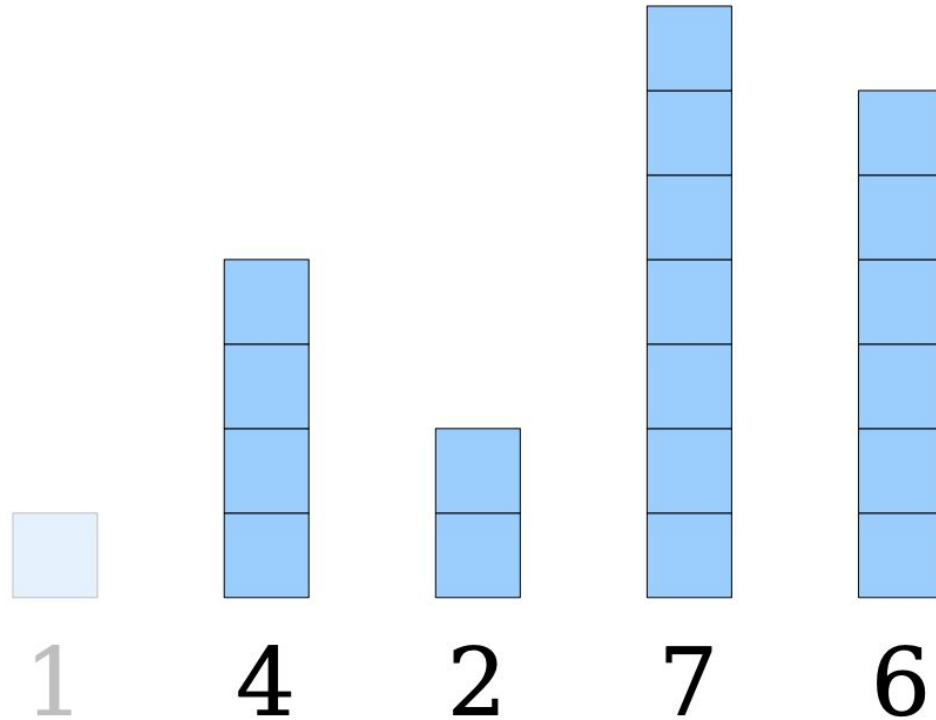
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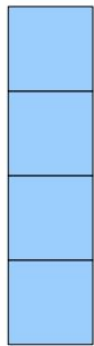


## Our first sort: Selection sort

*The smallest element of the remaining elements goes at the front of the remaining elements.*



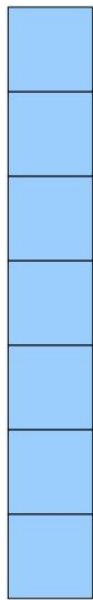
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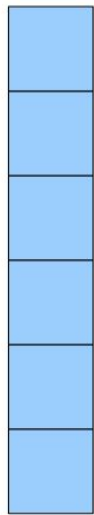
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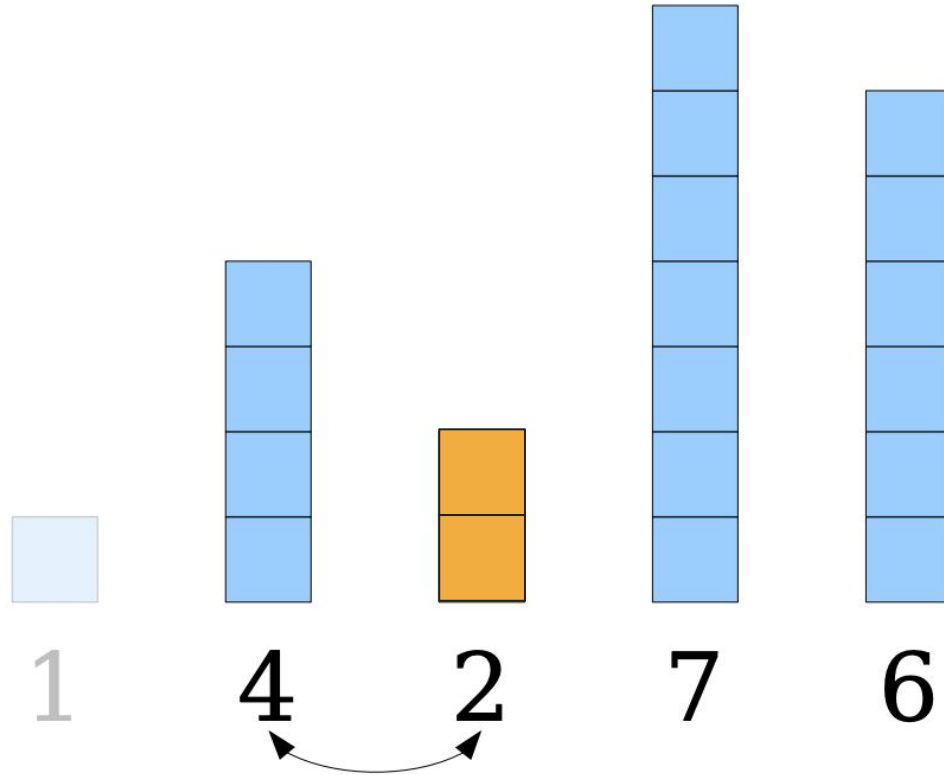


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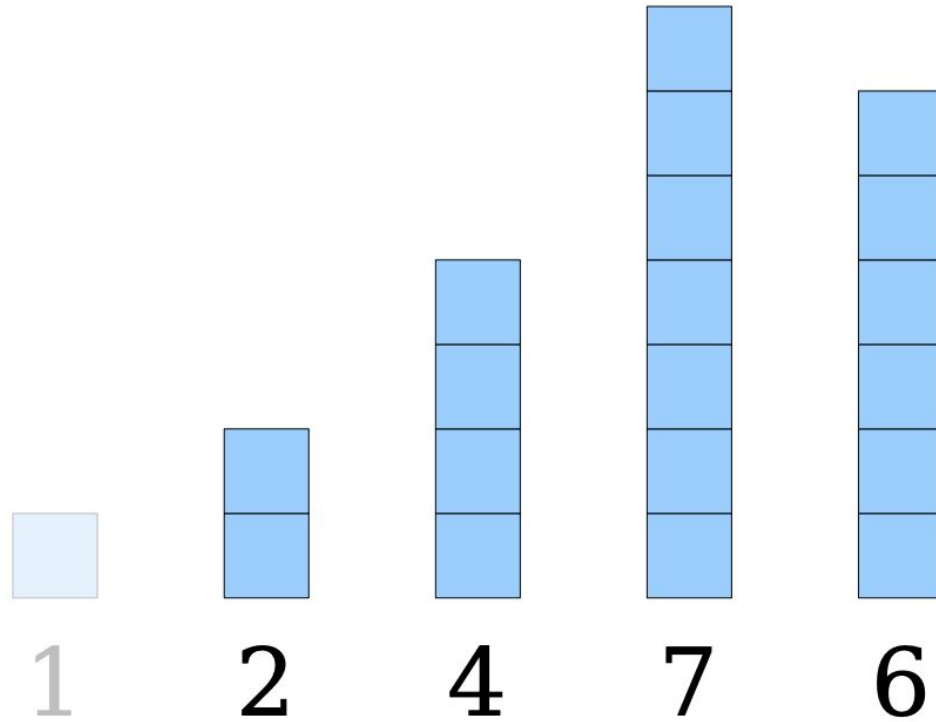


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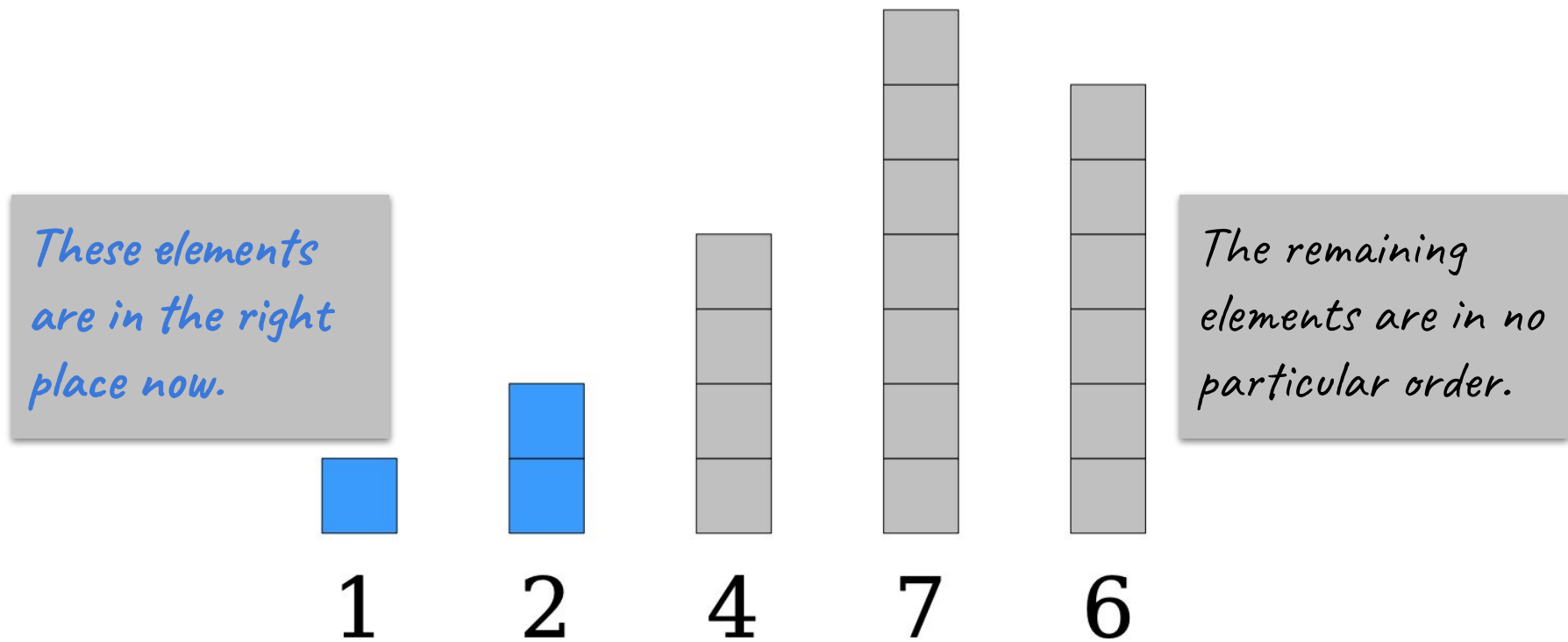
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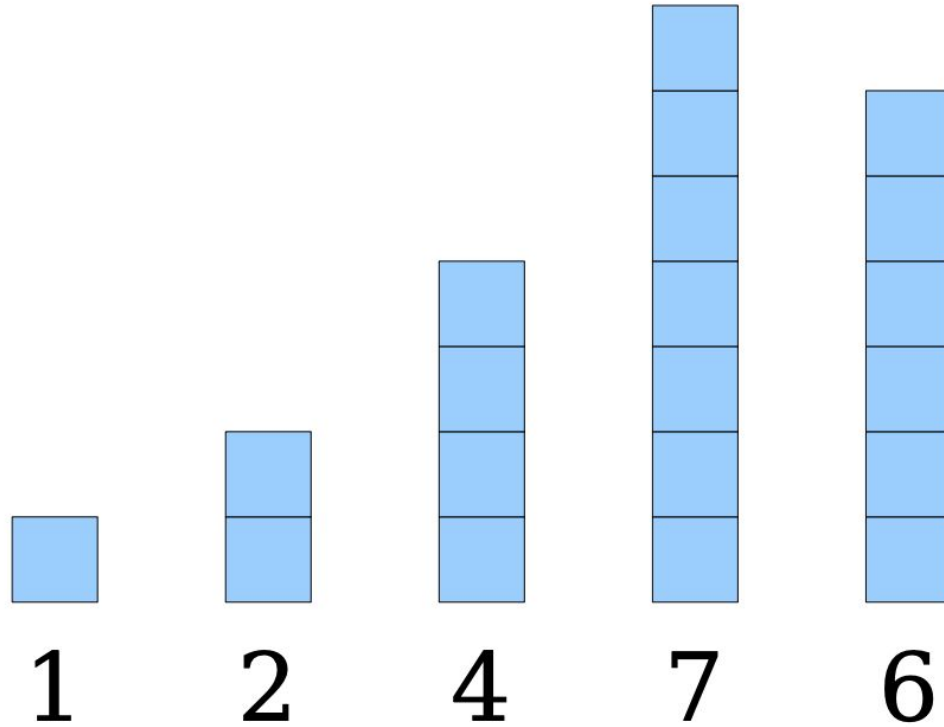
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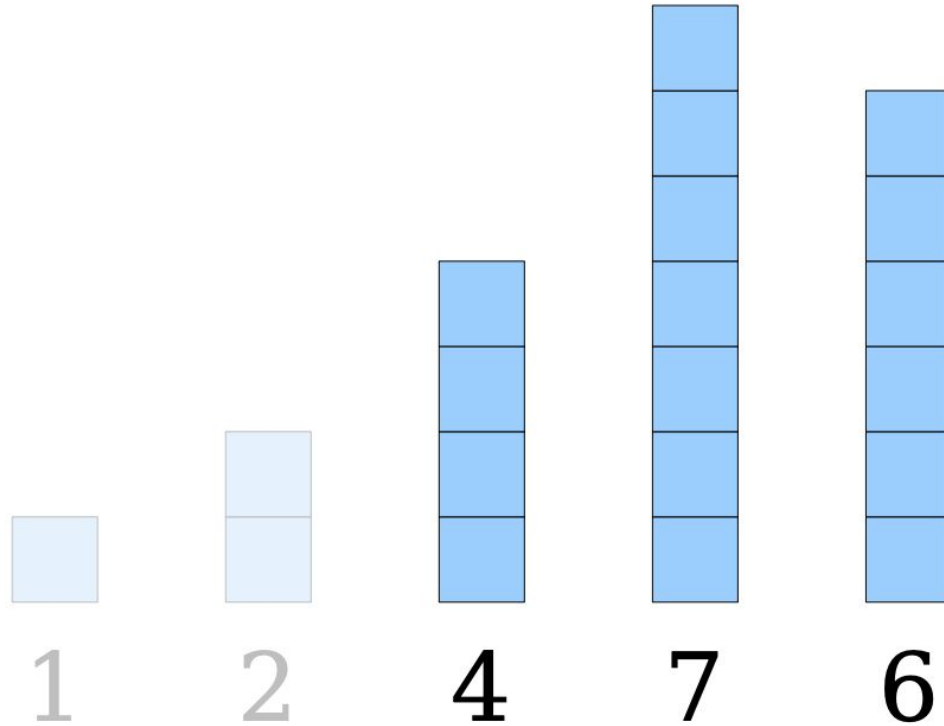
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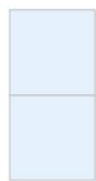


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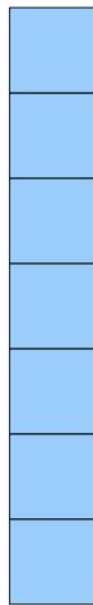
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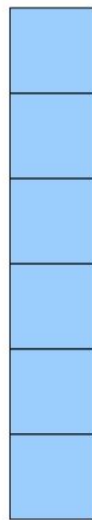
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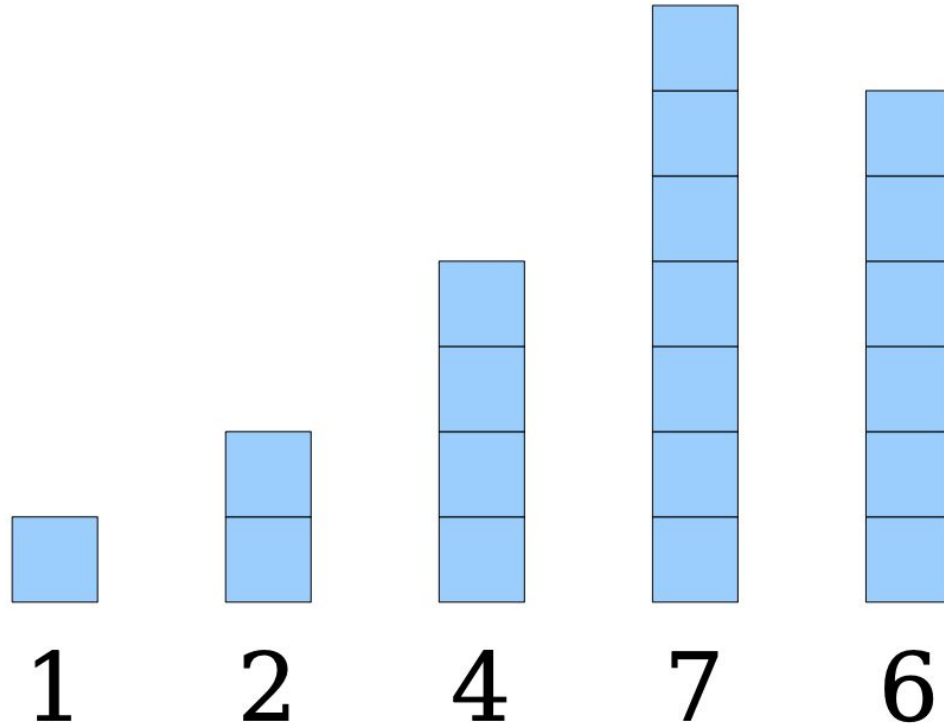


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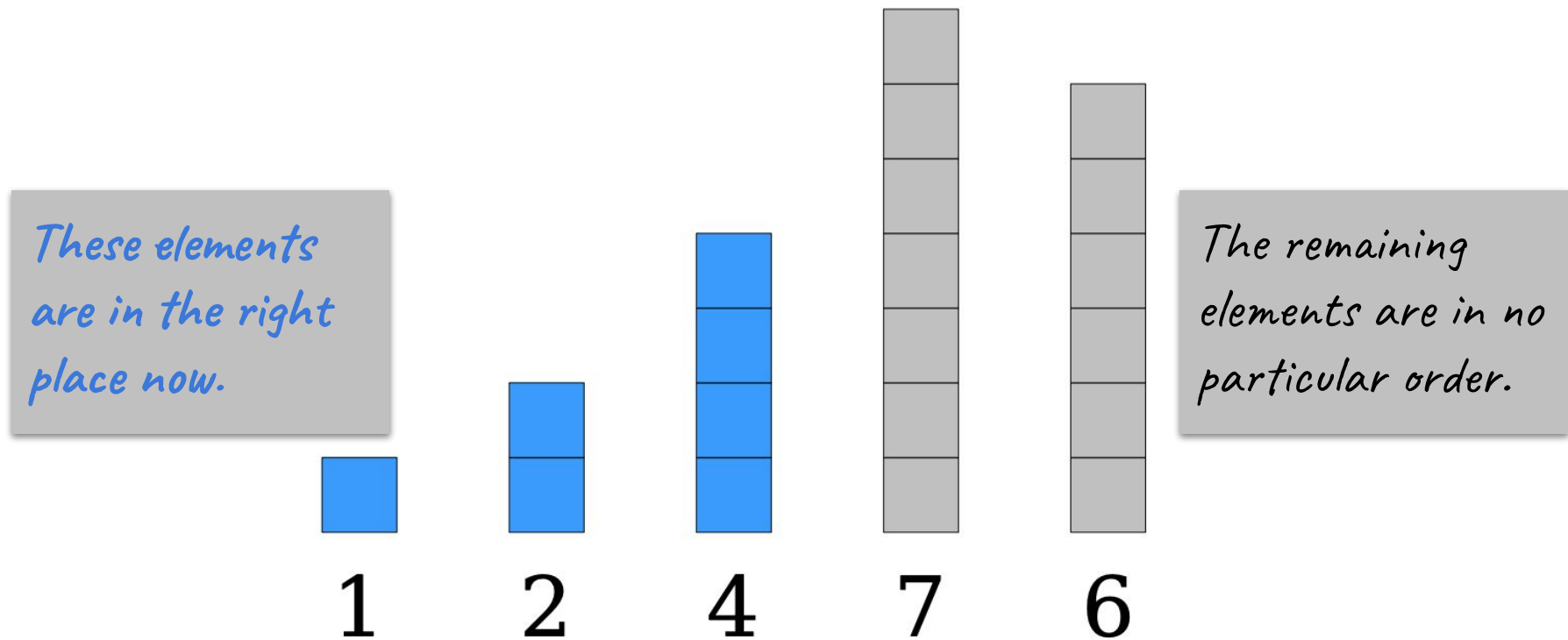
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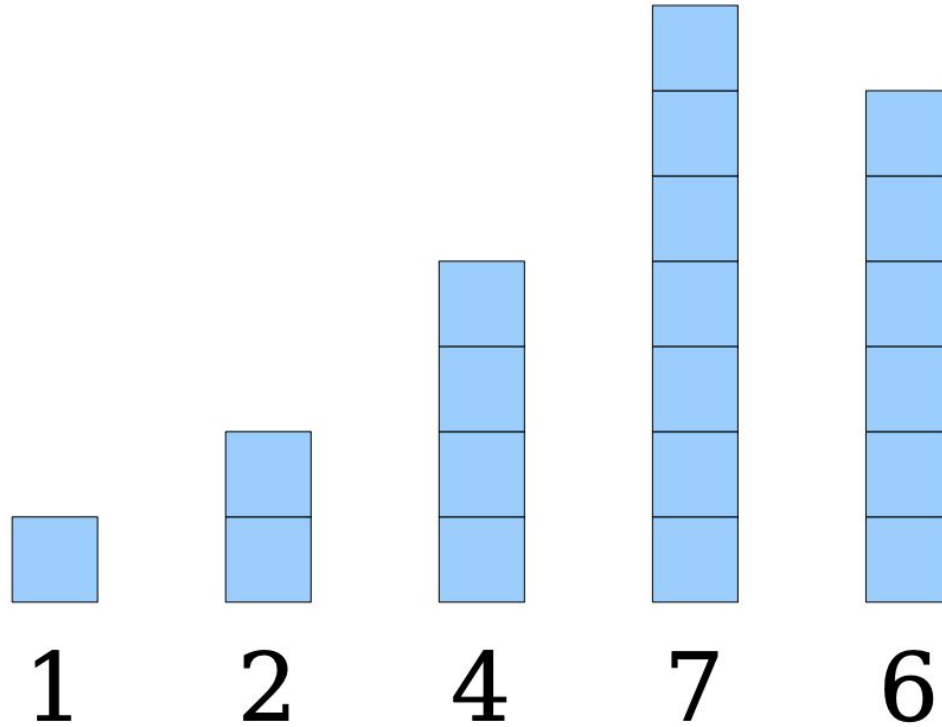




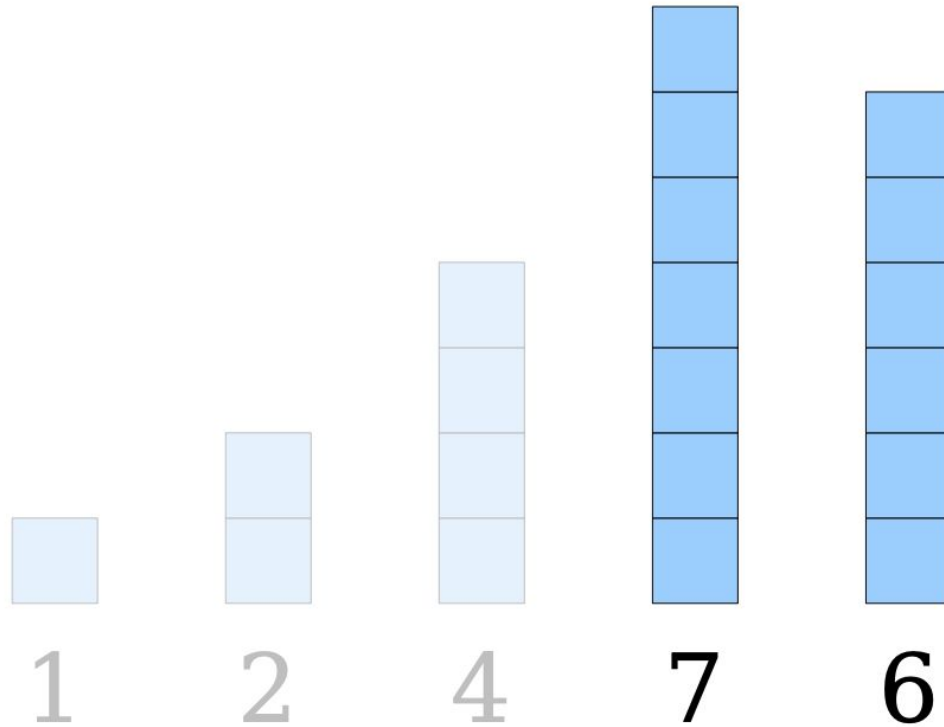
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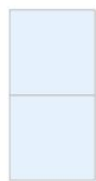


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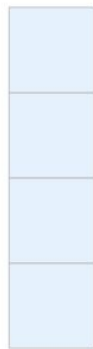
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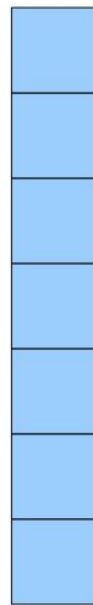
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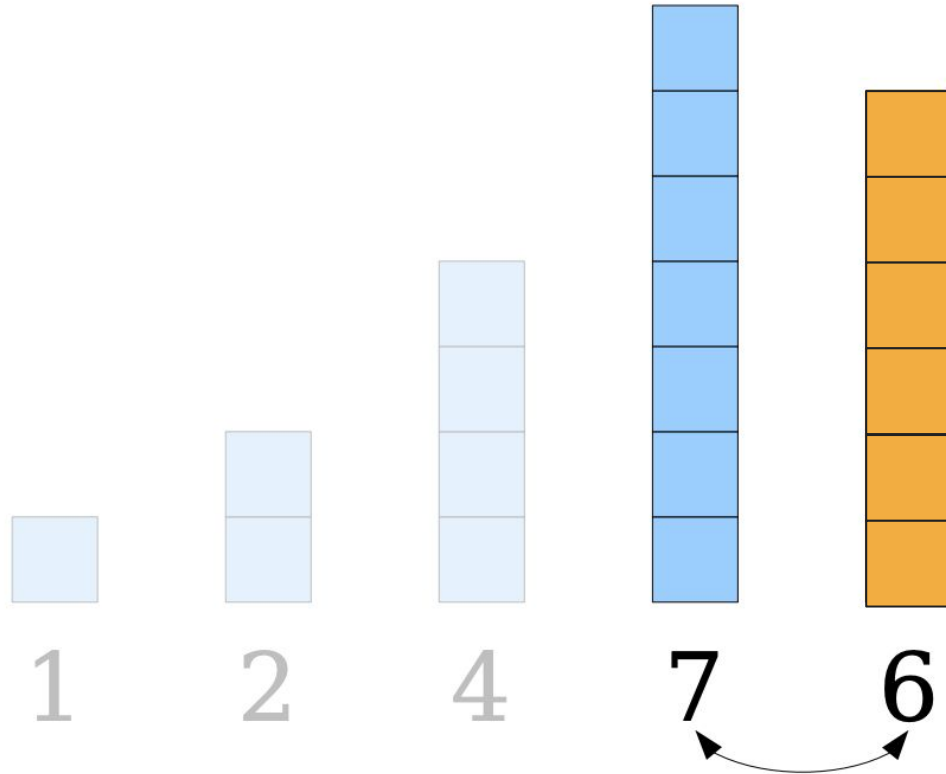


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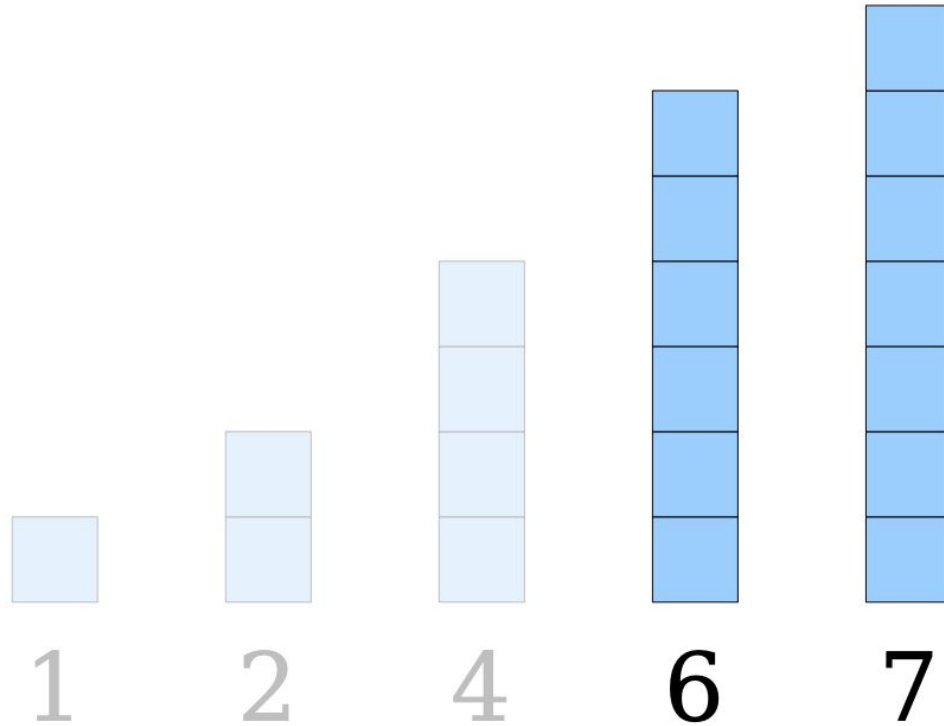


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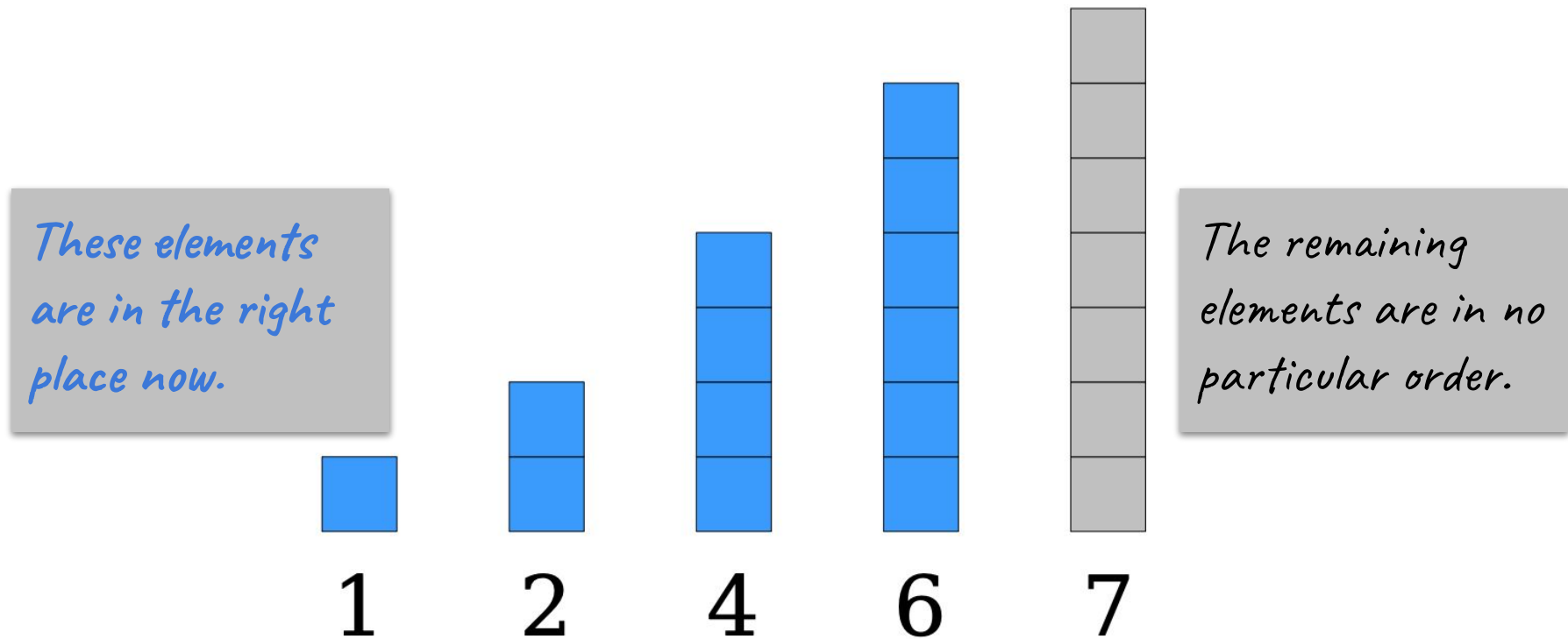
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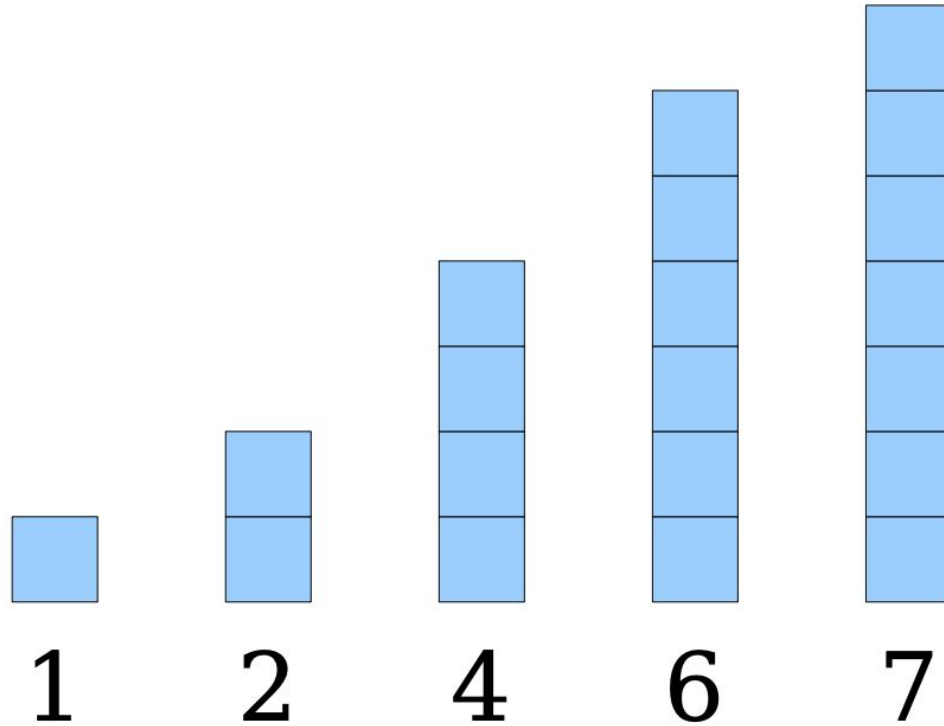
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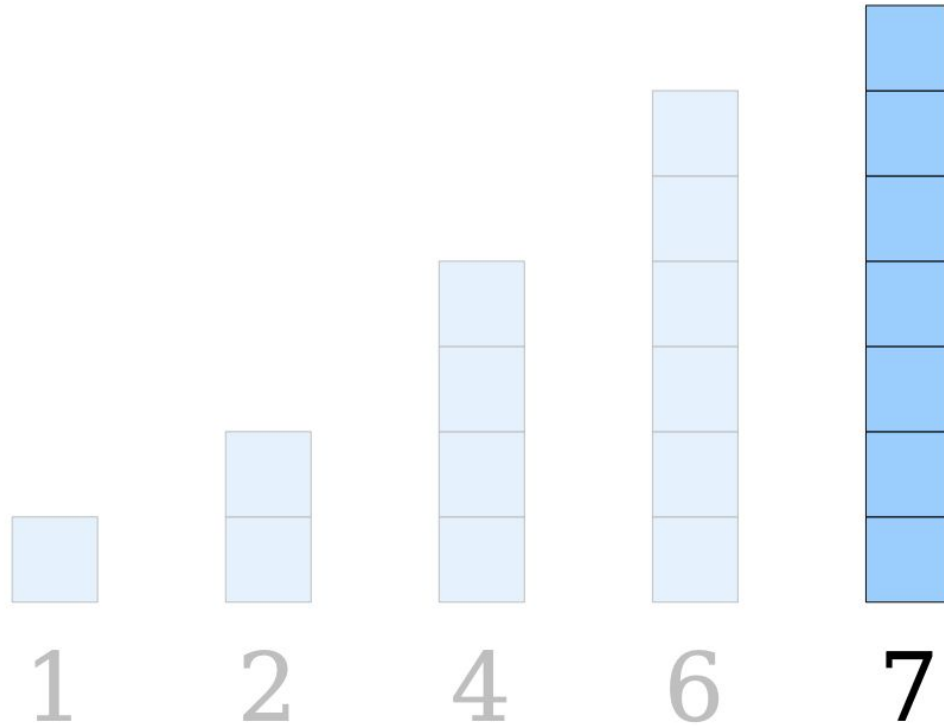


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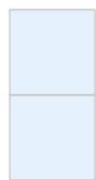


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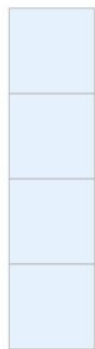
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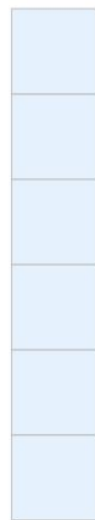
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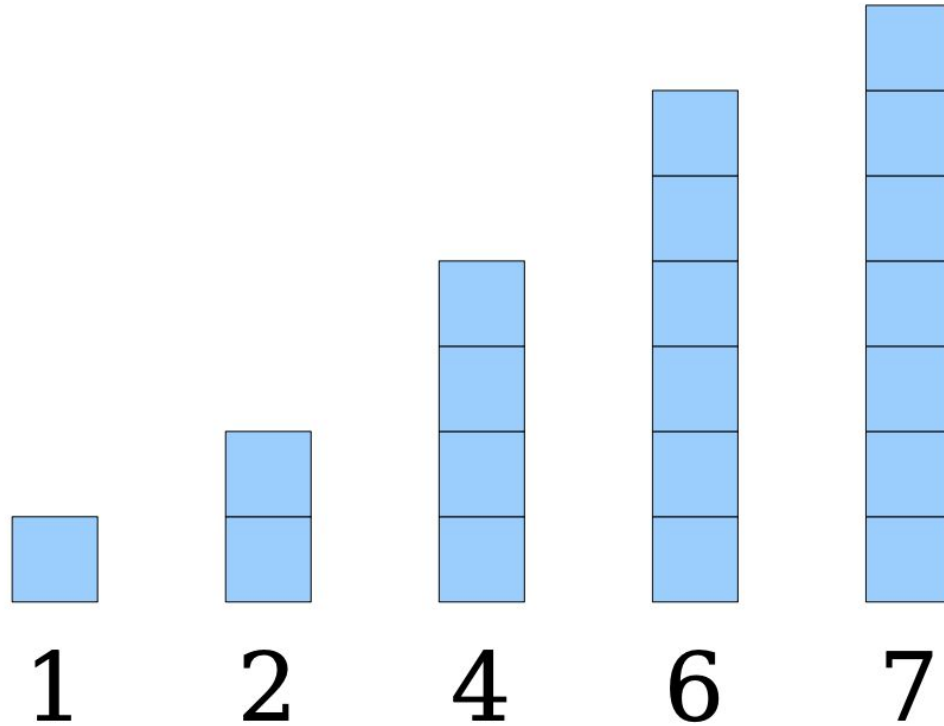


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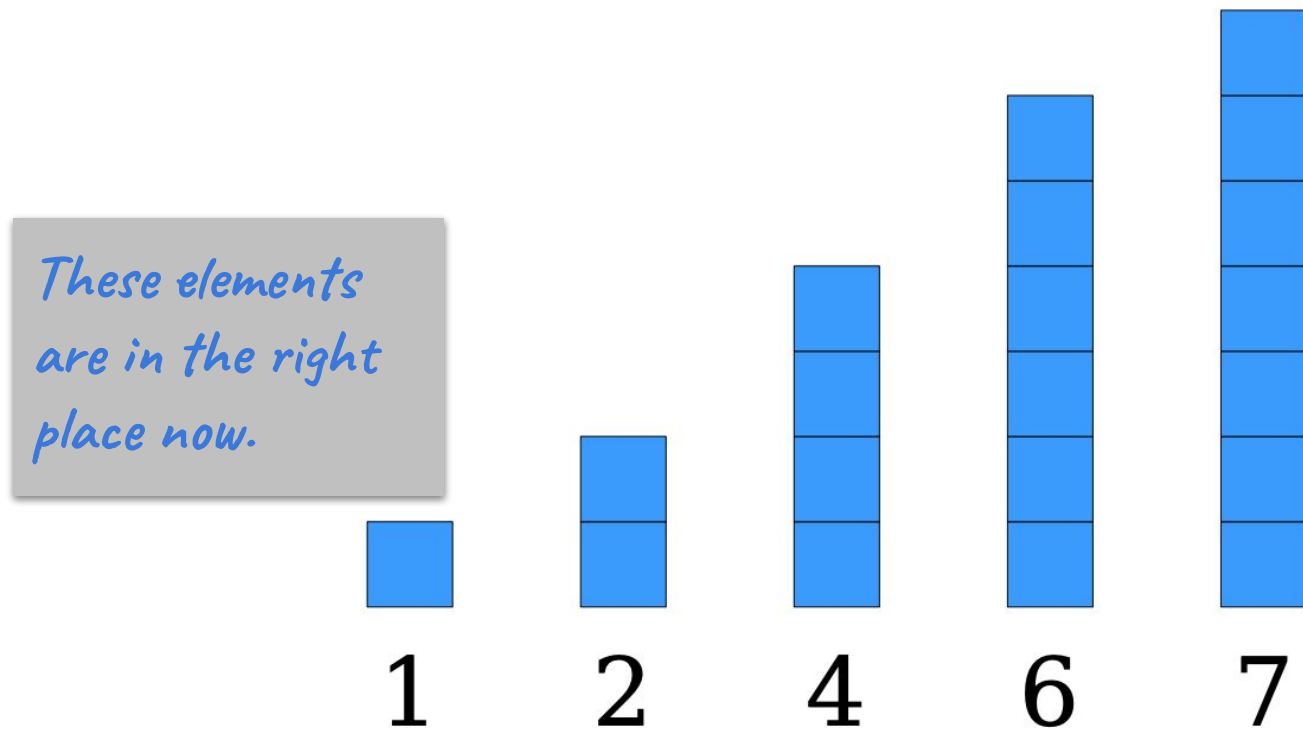


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# Our first sort: Selection sort



## Our first sort: Selection sort



# Selection sort algorithm

- Find the smallest element and move it to the first position.
- Find the smallest element of what's left and move it to the second position.
- Find the smallest element of what's left and move it to the third position.
- Find the smallest element of what's left and move it to the fourth position.
- (etc.)

```
void selectionSort(Vector<int>& elems) {
    for (int index = 0; index < elems.size(); index++) {
        int smallestIndex = indexOfSmallest(elems, index);
        swap(elems[index], elems[smallestIndex]);
    }
}
/**
 * Given a vector and a starting point, returns the index of the smallest
 * element in that vector at or after the starting point
 */
int indexOfSmallest(const Vector<int>& elems, int startPoint) {
    int smallestIndex = startPoint;
    for (int i = startPoint + 1; i < elems.size(); i++) {
        if (elems[i] < elems[smallestIndex]) {
            smallestIndex = i;
        }
    }
    return smallestIndex;
}
```

# Analyzing selection sort

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  - To find the third-smallest value, we need to look at  $n - 2$  elements.
  - This process continues until we have found every last "smallest element" from the original collection.
- This, the total amount of work we have to do is  $n + (n - 1) + (n - 2) + \dots + 1$

# The complexity of selection sort

- There is a mathematical formula that tells us

$$n + (n-1) + \dots + 2 + 1 = (n * (n+1)) / 2$$

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$$= O(n * (n+1)) \longleftarrow \text{Big-O ignores constant factors}$$

$$= O(n^2 + n)$$

$$= O(n^2) \longleftarrow \text{Big-O ignores low-order terms}$$

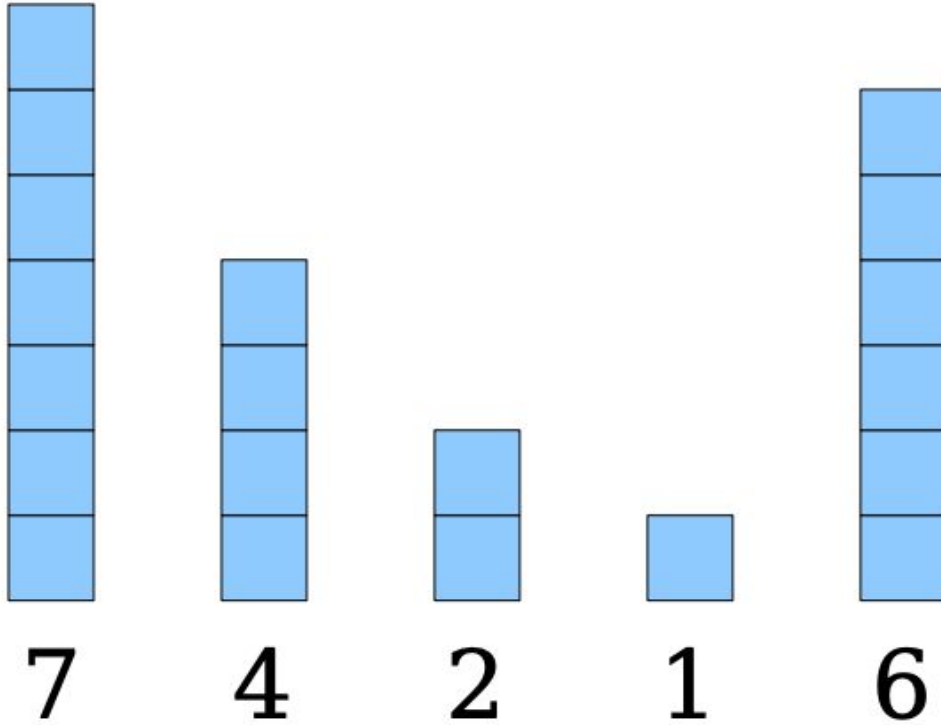
# Selection sort takeaways

- Selection sort works by "selecting" the smallest remaining element in the list and putting it in the front of all remaining elements.
- Selection sort is an  $O(n^2)$  algorithm.
- Can we do better?
  - Yes!

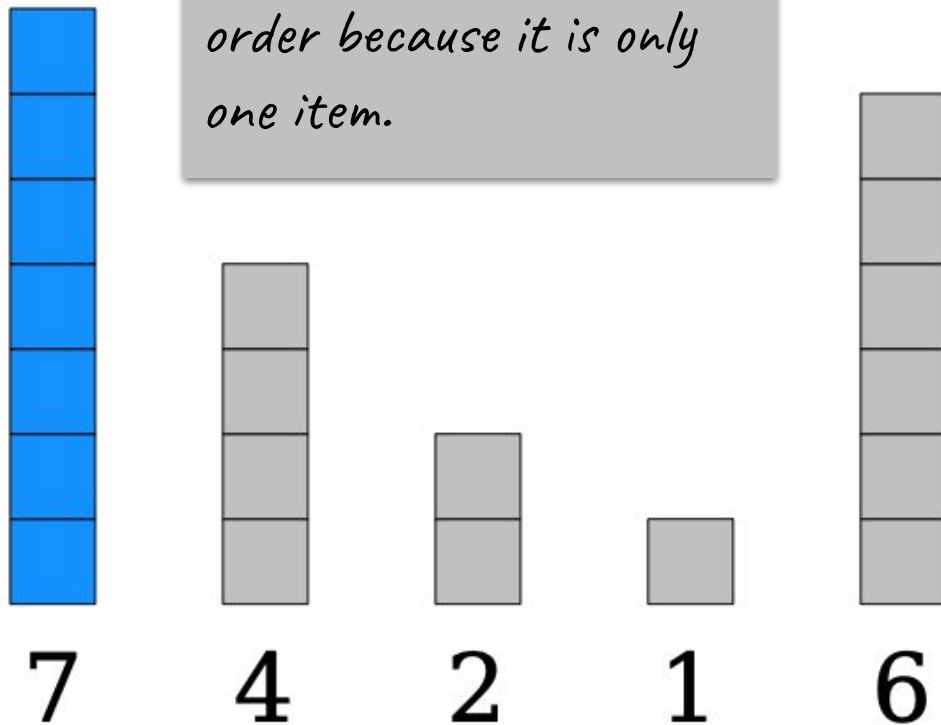
# Insertion Sort

(Bonus Content, not covered in live lecture)

# Insertion sort



# Insertion sort

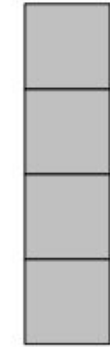


*Considered alone, the blue item is trivially in sorted order because it is only one item.*

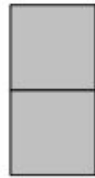
# Insertion sort



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*The items in gray are in no particular order (unsorted).*

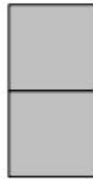
# Insertion sort



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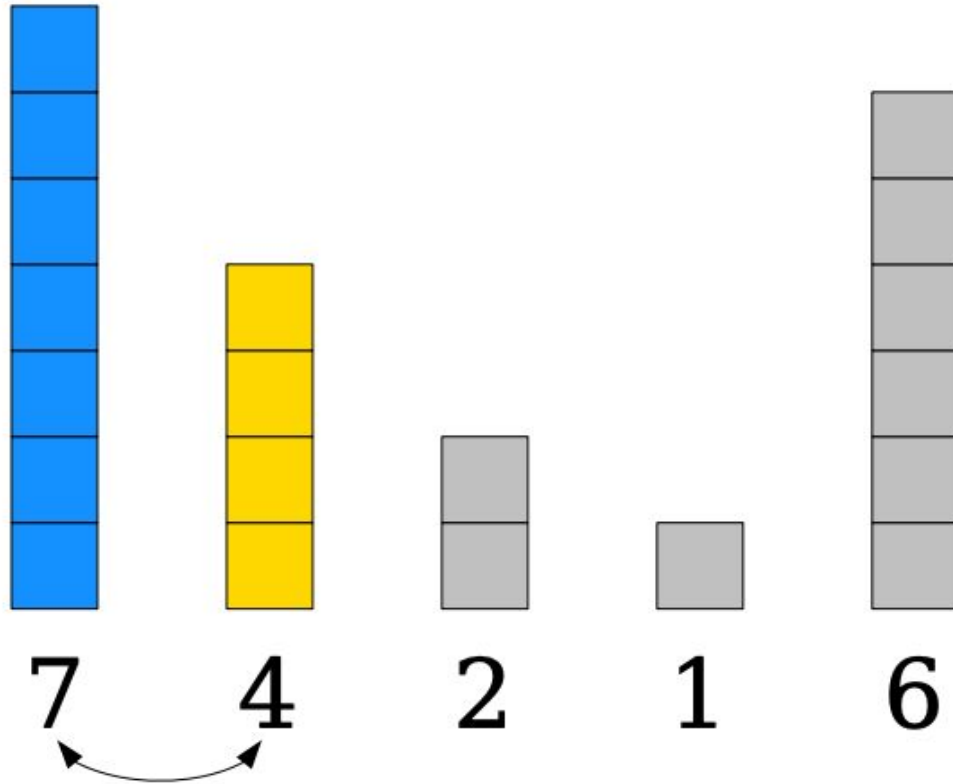
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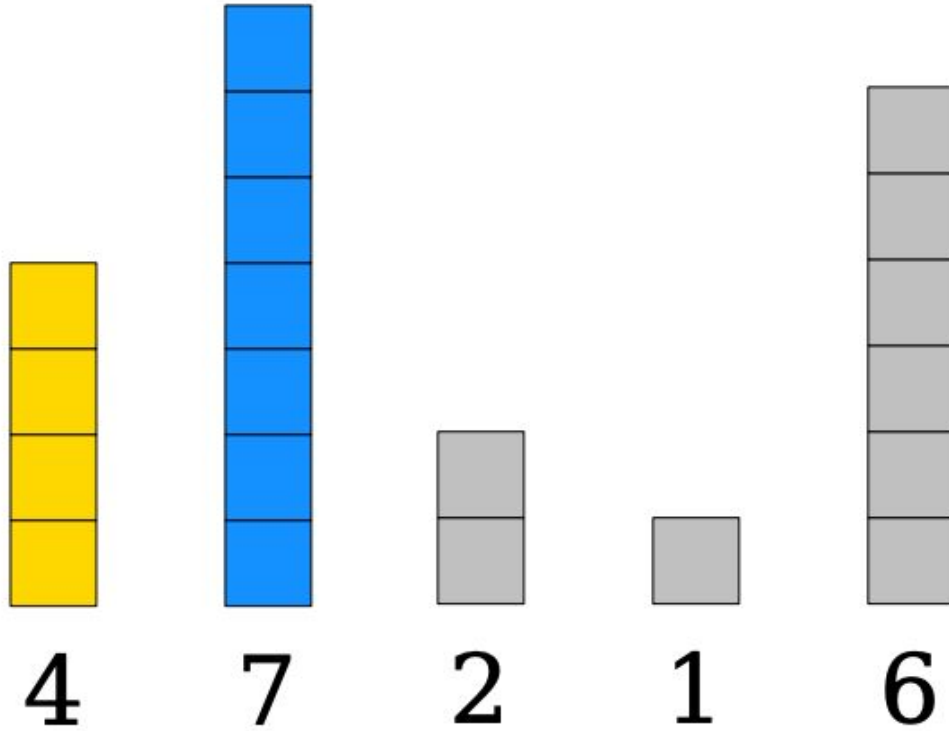
*Insert the yellow element into the sequence that includes the blue element.*

# Insertion sort

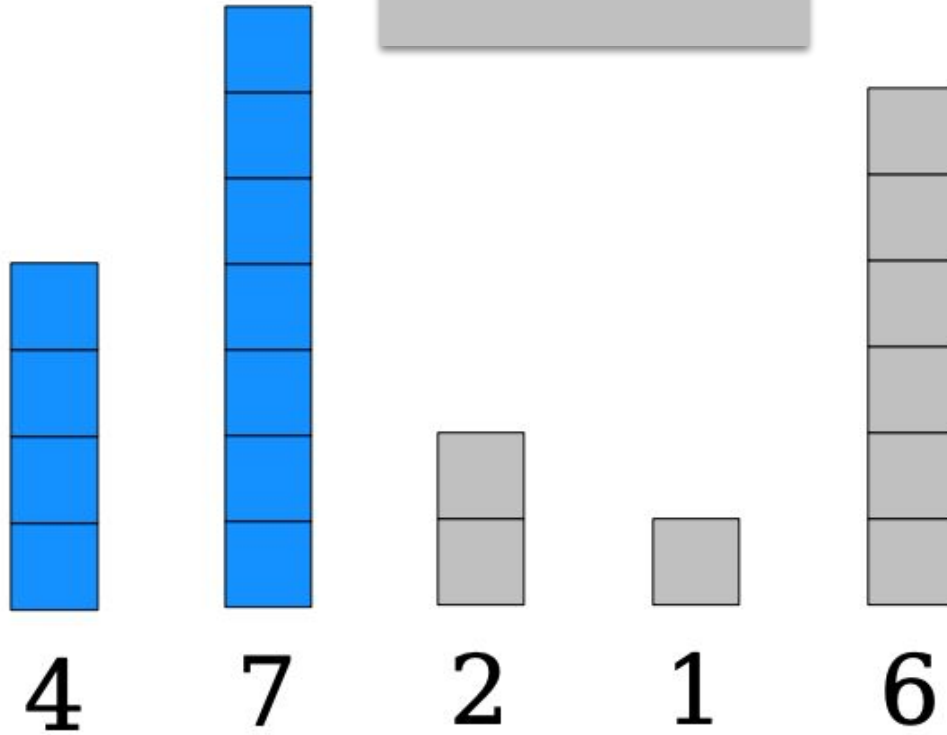




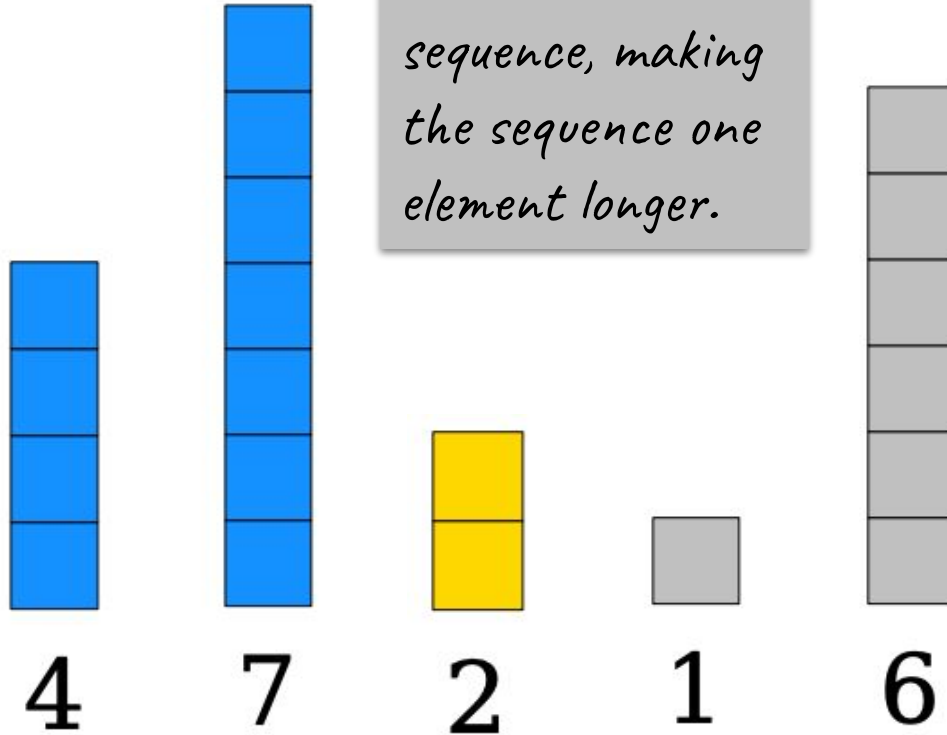
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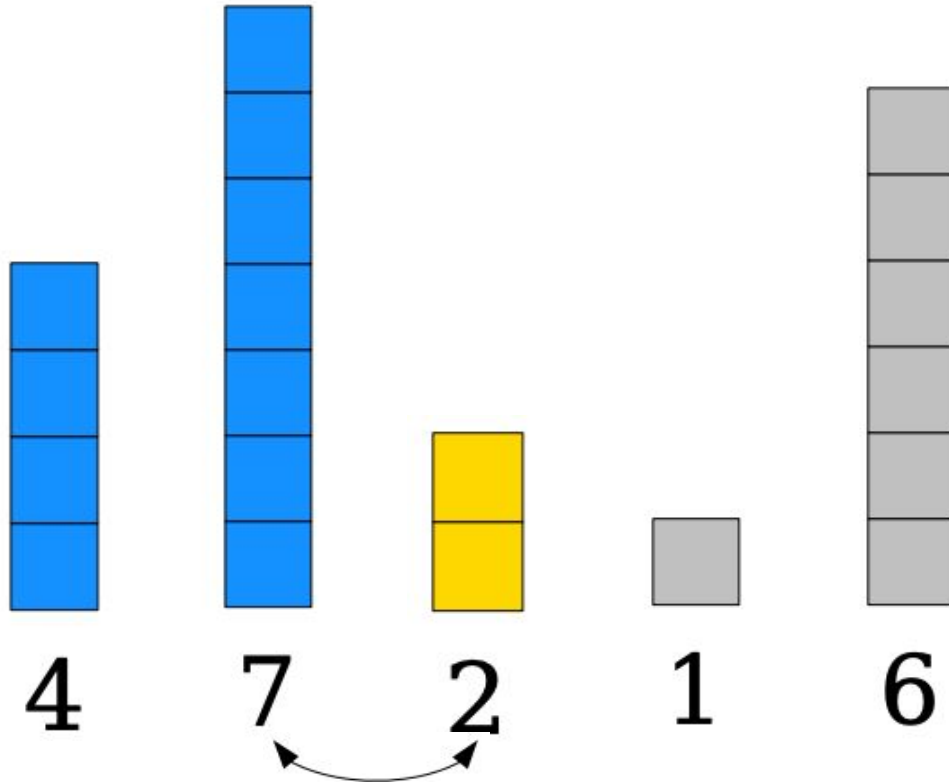
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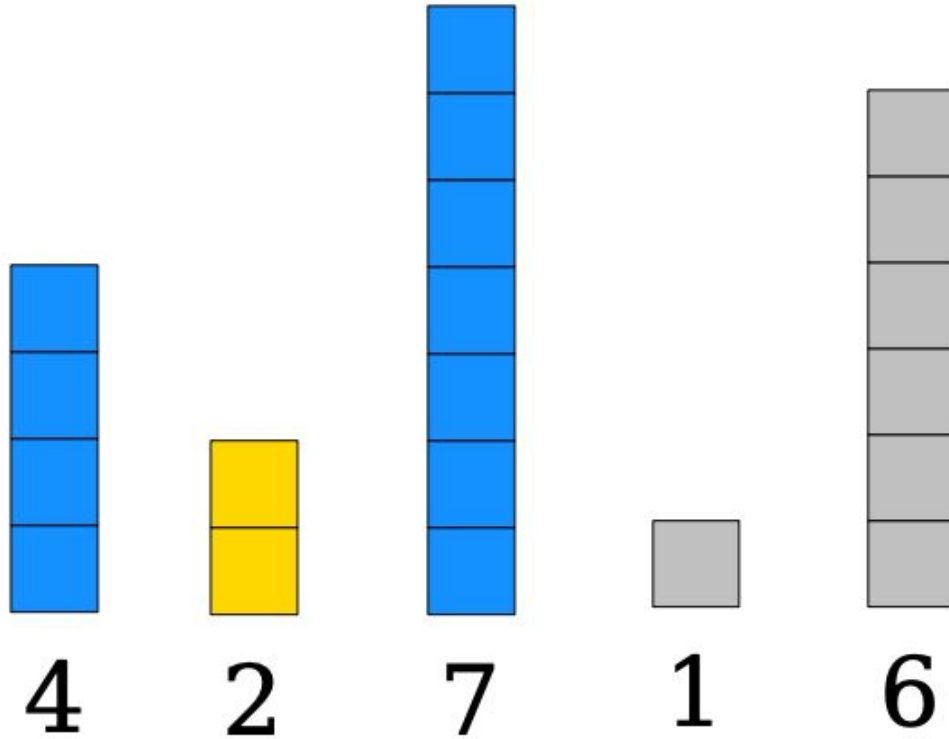
# Insertion sort



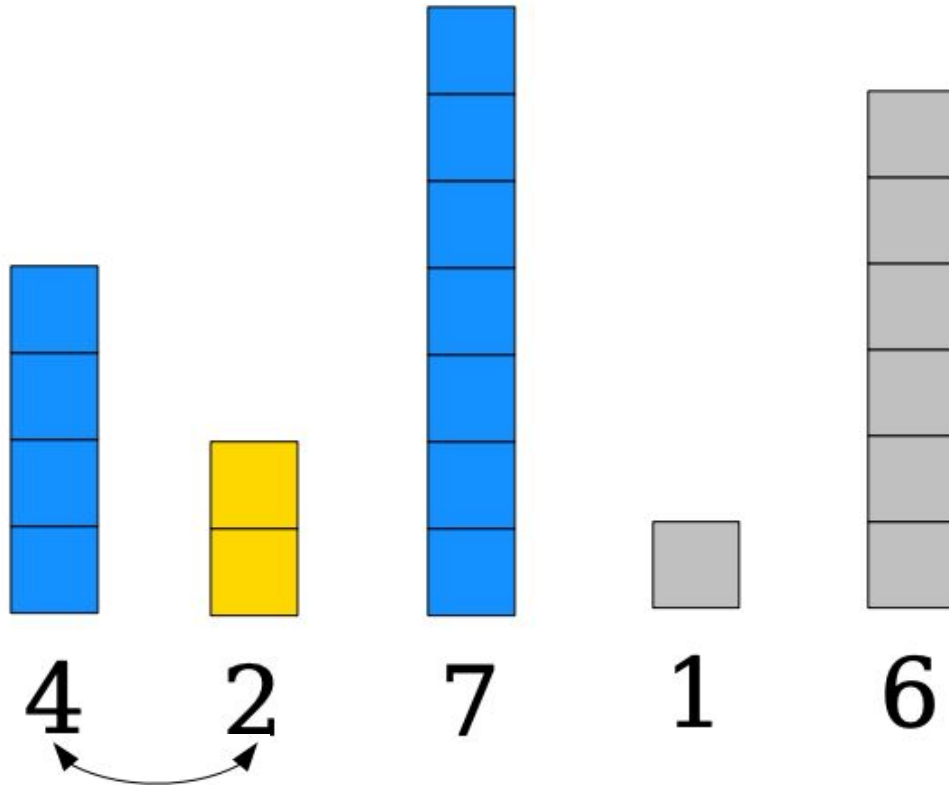
# Insertion sort



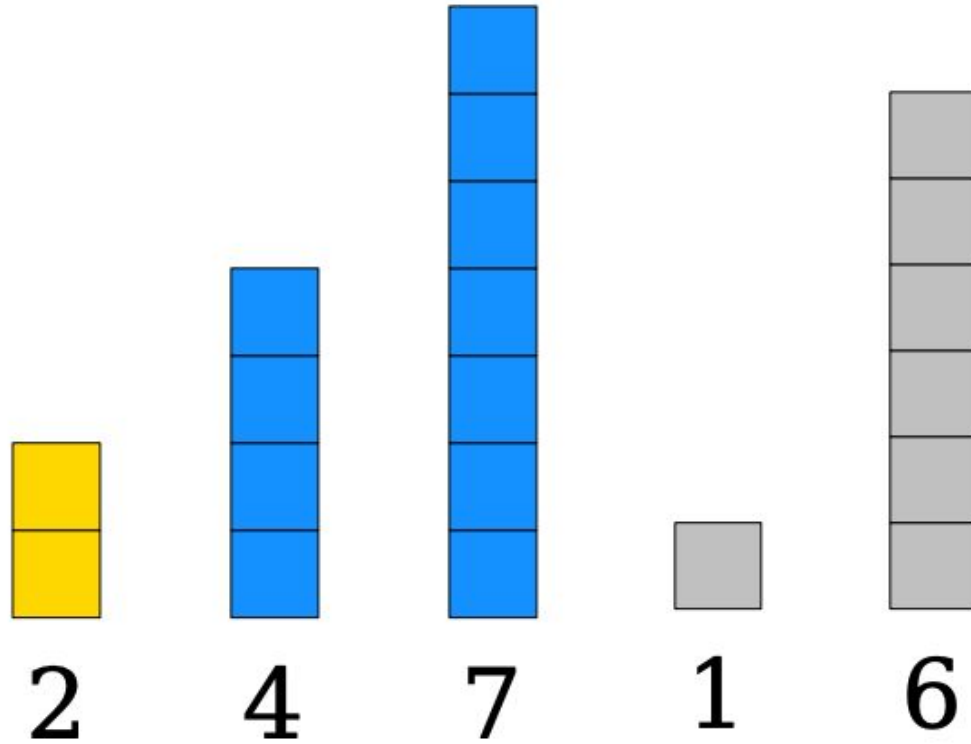
# Insertion sort



# Insertion sort

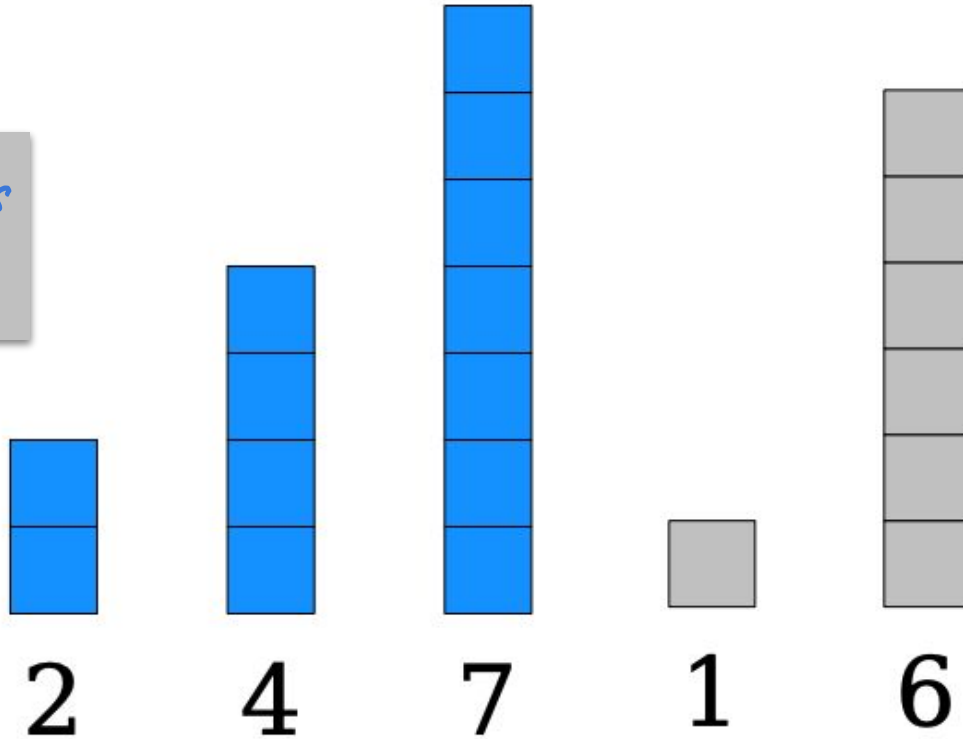


# Insertion sort



# Insertion sort

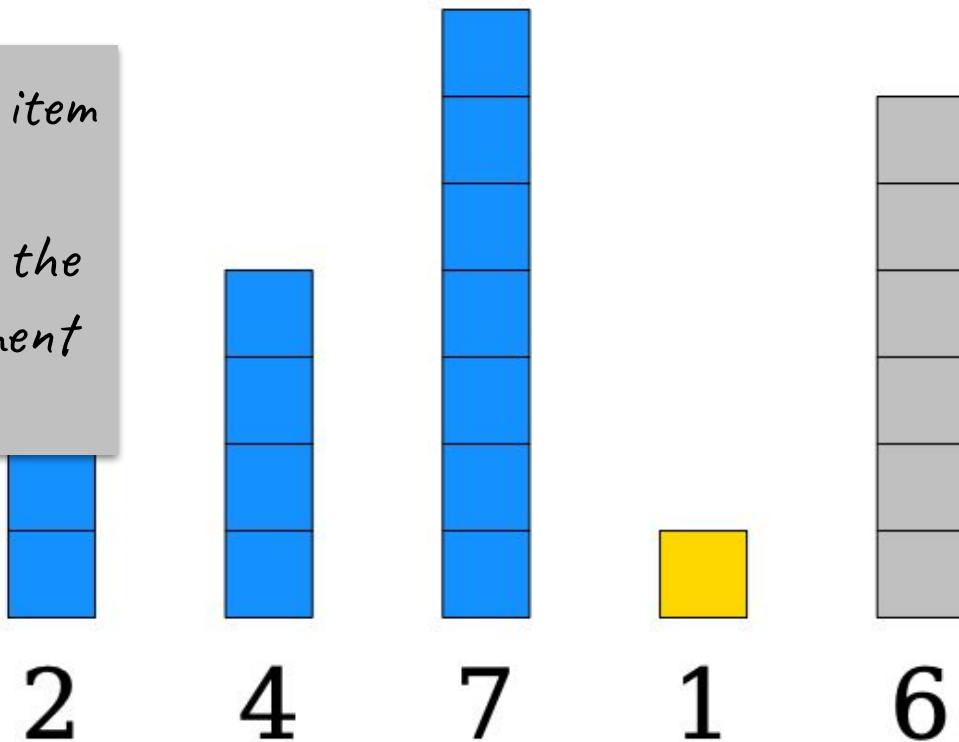
*The blue elements  
are sorted!*



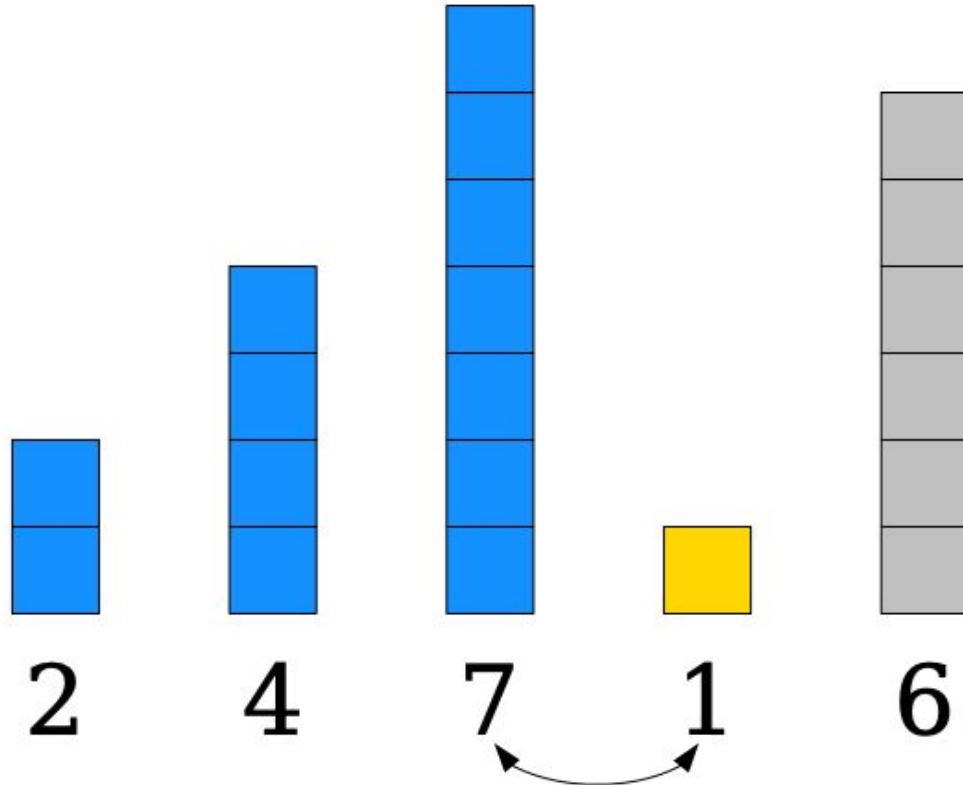


# Insertion sort

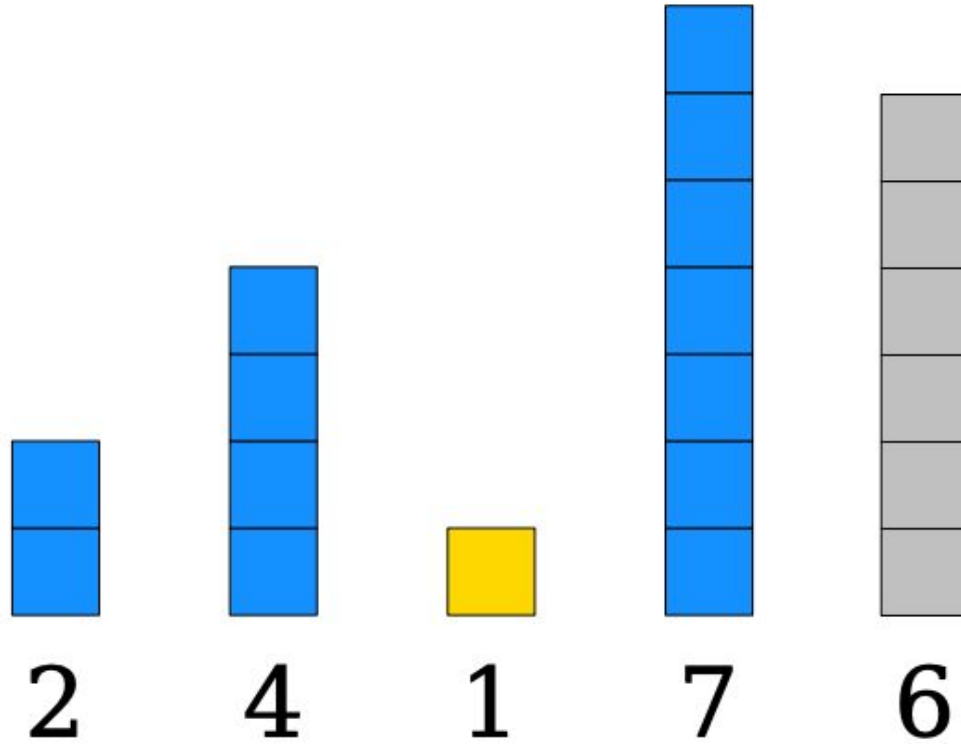
*Insert the yellow item into the blue sequence, making the sequence one element longer.*



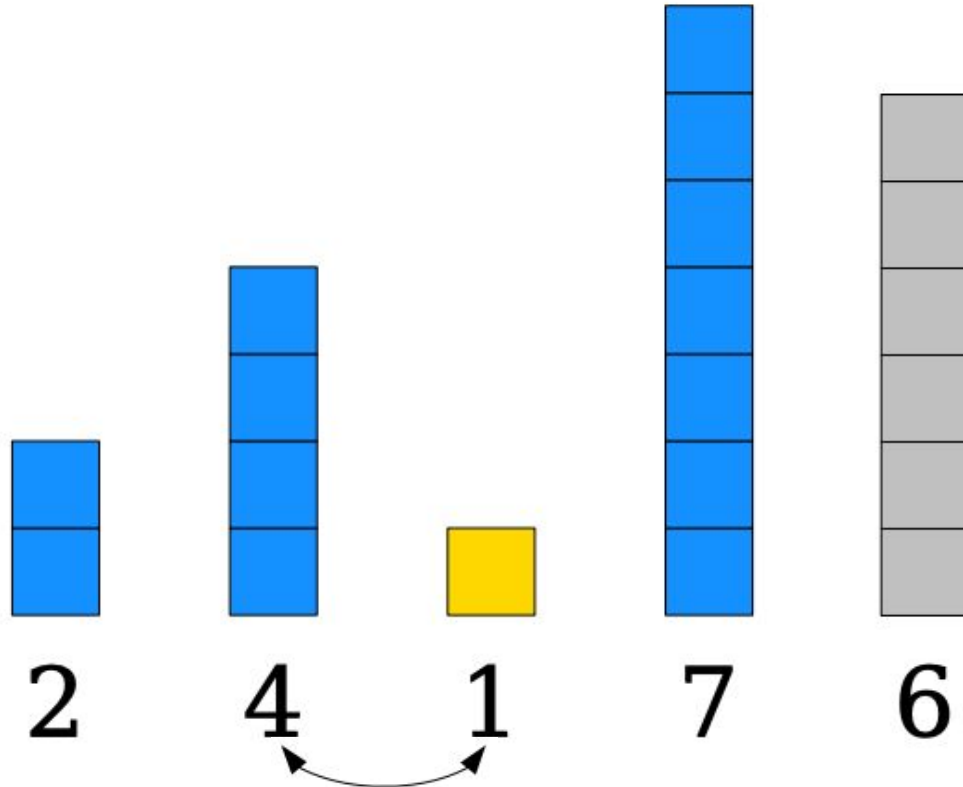
# Insertion sort



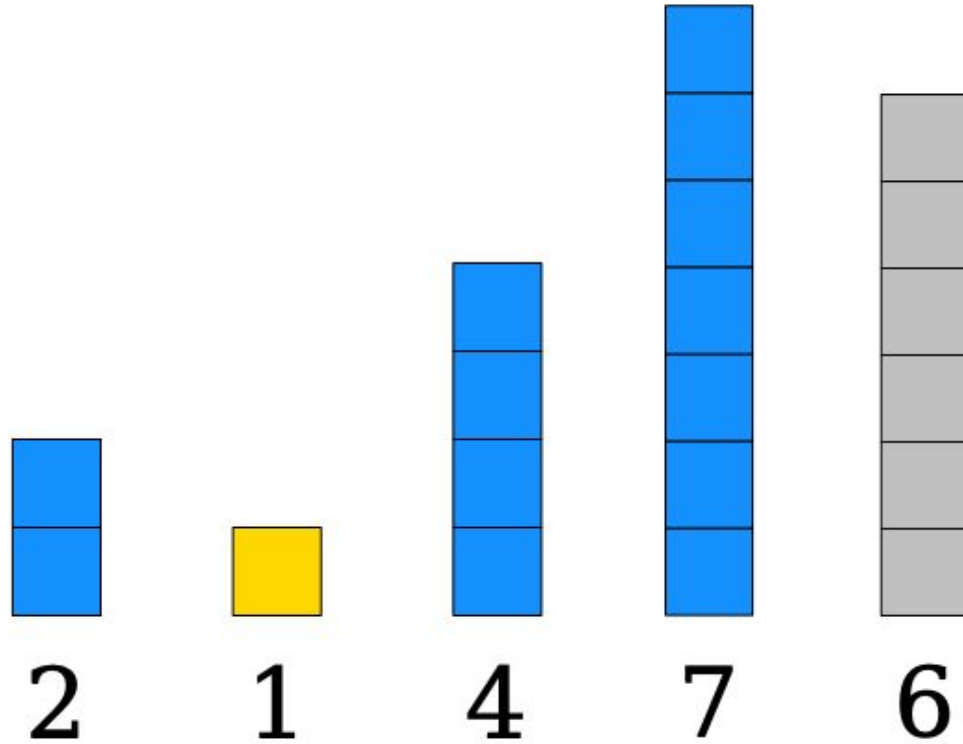
# Insertion sort



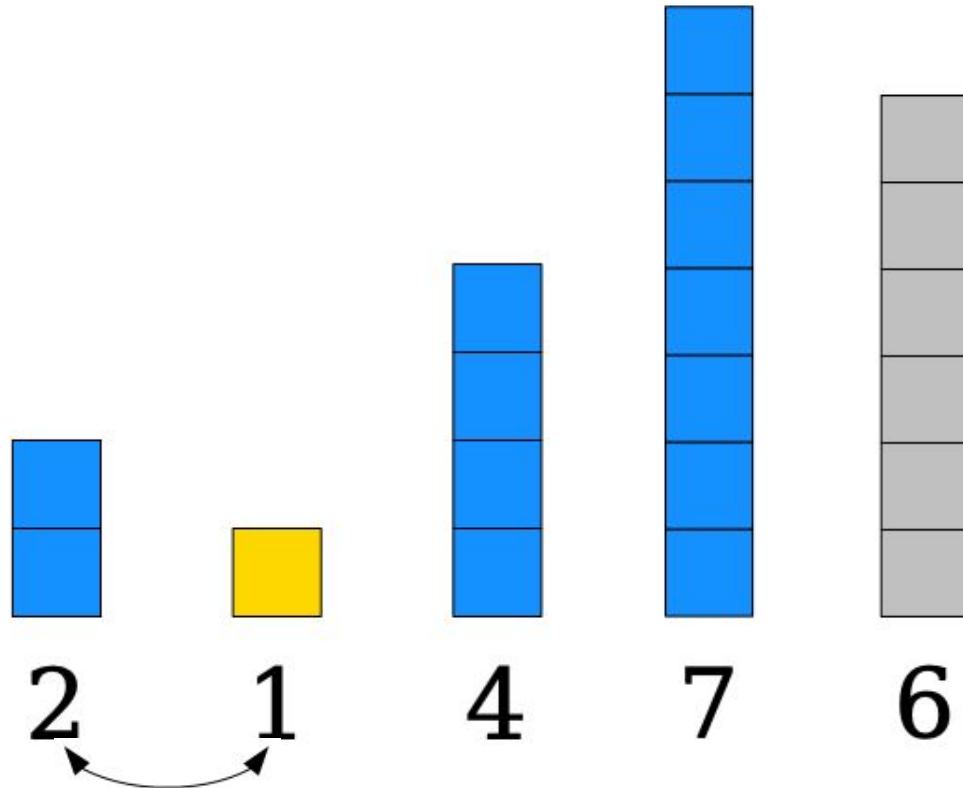
# Insertion sort



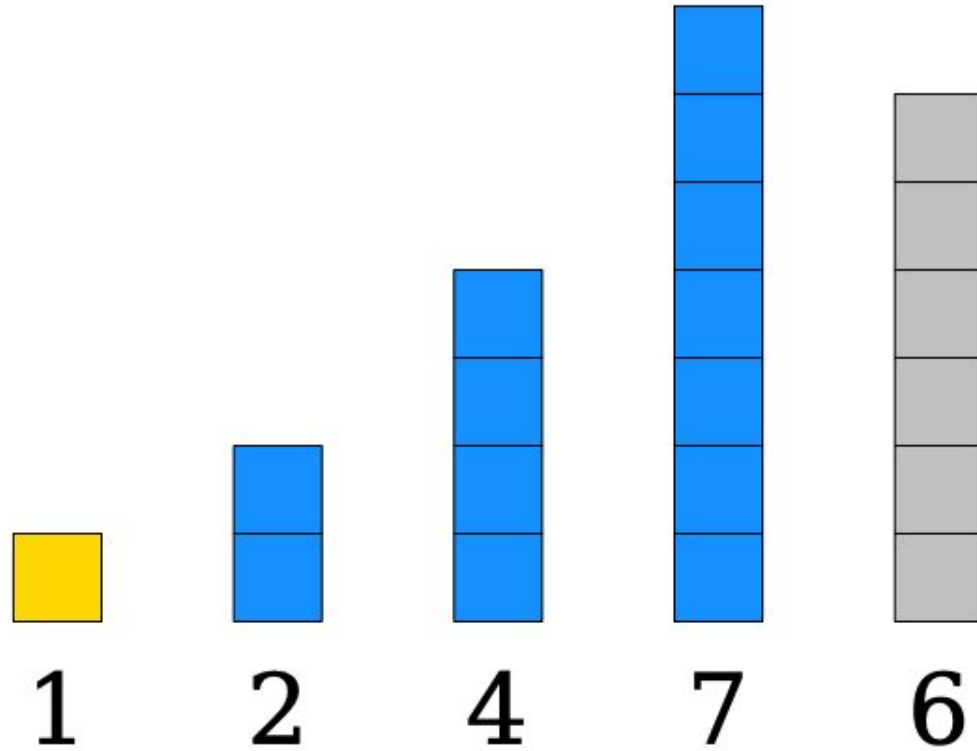
# Insertion sort



# Insertion sort

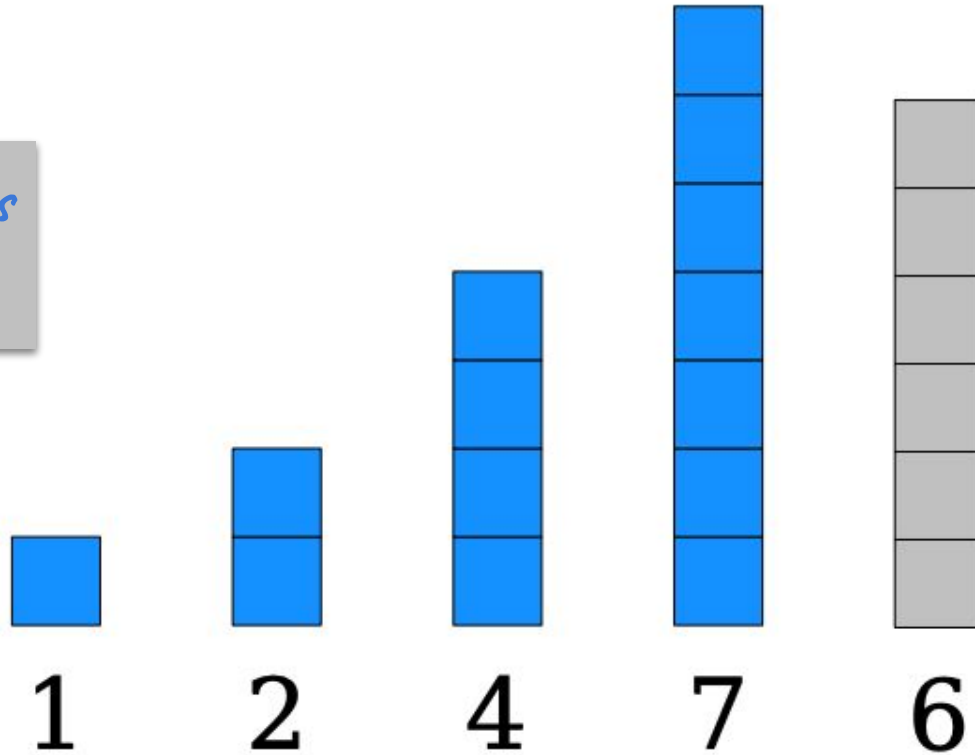


# Insertion sort



# Insertion sort

*The blue elements  
are sorted!*





# Insertion sort

*Insert the yellow item into the blue sequence, making the sequence one element longer.*



1



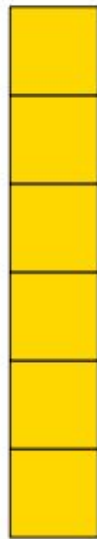
2



4

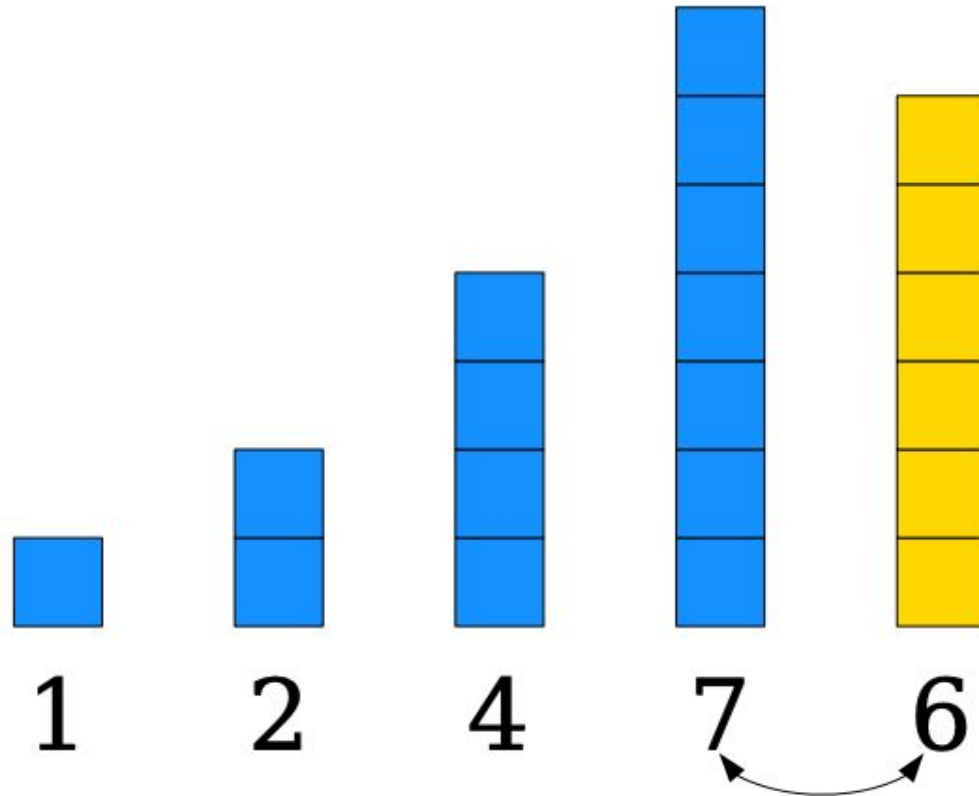


7

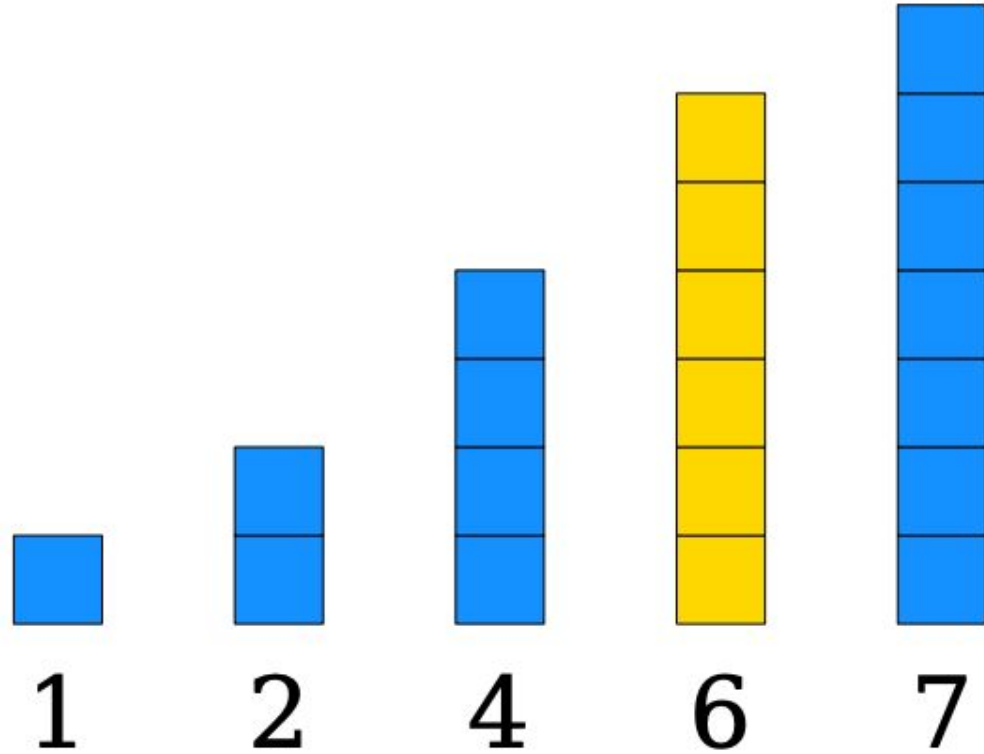


6

# Insertion sort

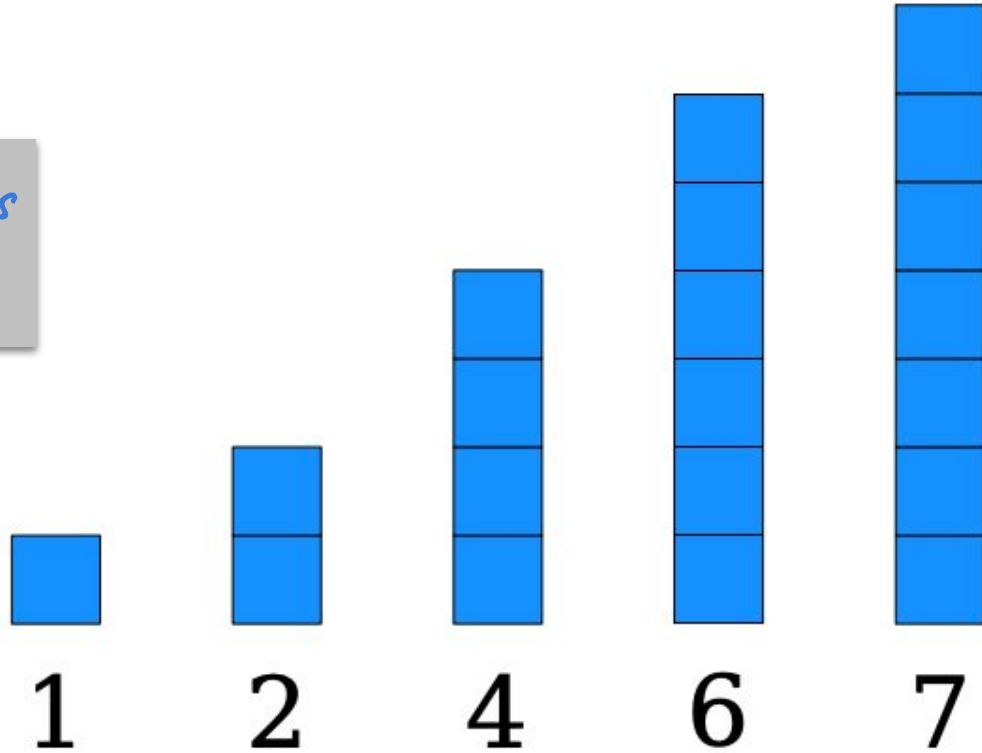


# Insertion sort



# Insertion sort

*The blue elements  
are sorted!*



# Insertion sort algorithm

- Repeatedly insert an element into a sorted sequence at the front of the array.
- To insert an element, swap it backwards until either:
  - (1) it's bigger than the element before it, or
  - (2) it's at the front of the array.

# Insertion sort code

```
void insertionSort(Vector<int>& v) {
    for (int i = 0; i < v.size(); i++) {
        /* Scan backwards until either (1) the preceding
         * element is no bigger than us or (2) there is
         * no preceding element. */
        for (int j = i - 1; j >= 0; j--) {
            if (v[j] <= v[j + 1]) break;
            /* Swap this element back one step. */
            swap(v[j], v[j + 1]);
        }
    }
}
```

# The complexity of insertion sort

- In the worst case (the array is in reverse sorted order), insertion sort takes time  $O(n^2)$ .
  - The analysis for this is similar to selection sort!
- In the best case (the array is already sorted), insertion takes time  $O(n)$  because you only iterate through once to check each element.
  - Selection sort, however, is always  $O(n^2)$  because you always have to search the remainder of the list to guarantee that you're finding the minimum at each step.
- **Fun fact:** Insertion sorting an array of random values takes, *on average*,  $O(n^2)$  time.
  - This is beyond the scope of the class – take CS109 if you're interested in learning more!

How can we design better,  
more efficient sorting  
algorithms?



# Divide-and-Conquer

# Motivating Divide-and-Conquer

- So far, we've seen  $O(N^2)$  sorting algorithms. How can we start to do better?

# Motivating Divide-and-Conquer

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- Assume that it takes  $t$  seconds to run selection sort on the following array:

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4
----	---	---	---	---	----	---	----	---	----	---	----	---	----	----	---

# Motivating Divide-and-Conquer

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----	---	---	---	---	----	---	----	---	----	---	----	---	----	----	---

- Poll: Approximately how many seconds will it take to run selection sort on **each** of the following arrays? [pollev.com/cs106bpoll](http://pollev.com/cs106bpoll)

14	6	3	9	7	16	2	15
----	---	---	---	---	----	---	----

5	10	8	11	1	13	12	4
---	----	---	----	---	----	----	---

# How long would it take to sort an array half the size?

Same time ( $t$  seconds)

Half the time ( $t/2$  seconds)

A quarter of the time ( $t/4$  seconds)

An eighth of the time ( $t/8$  seconds)

None of the above



# Motivating Divide-and-Conquer

- So far, we've seen  $O(N^2)$  sorting algorithms. How can we start to do better?

- Assume that it takes *Answer: Each array should only take about  $t/4$  seconds to sort.* the following array:



- Poll: Approximately how long will it take to sort each of the following arrays? in selection sort on **each**



# Motivating Divide-and-Conquer

- Main insight:
  - Sorting  $\mathbf{N}$  elements directly takes total time  $\mathbf{t}$
  - Sorting two sets of  $\mathbf{N}/2$  elements (total of  $\mathbf{N}$  elements) takes  $\mathbf{t}/4 + \mathbf{t}/4 = \mathbf{t}/2$
  - We got a speedup just by sorting smaller sets of elements at a time!

# Motivating Divide-and-Conquer

- Main insight:
  - Sorting  $N$  elements directly takes total time  $t$
  - Sorting two sets of  $N/2$  elements (total of  $N$  elements) takes total time  $t/2$
  - We got a speedup just by sorting smaller sets of elements at a time!
- The main idea behind divide-and-conquer algorithms takes advantage of this. Let's design algorithms that break up a problem into many smaller problems that can be solved in parallel!



# General Divide-and-Conquer Approach

- Our general approach when designing a divide-and-conquer algorithm is to decide how to make the problem smaller and how to unify the results of these solved, smaller problems.

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- Our general approach when designing a divide-and-conquer algorithm is to decide how to make the problem smaller and how to unify the results of these solved, smaller problems.
- Both sorting algorithms we explore today will have both of these components:
  - Divide Step
    - Make the problem smaller by splitting up the input list
  - Join Step
    - Unify the newly sorted sublists to build up the overall sorted result

# General Divide-and-Conquer Approach

- Our general approach when designing a divide-and-conquer algorithm is to decide how to make the problem smaller and how to unify the results of these solved, smaller problems.
- Both sorting algorithms we explore today will have both of these components:
  - Divide Step
    - Make the problem smaller by splitting up the input list
  - Join Step
    - Unify the newly sorted sublists to build up the overall sorted result
- Divide-and-Conquer is a ripe time to return to recursion!

# Merge Sort

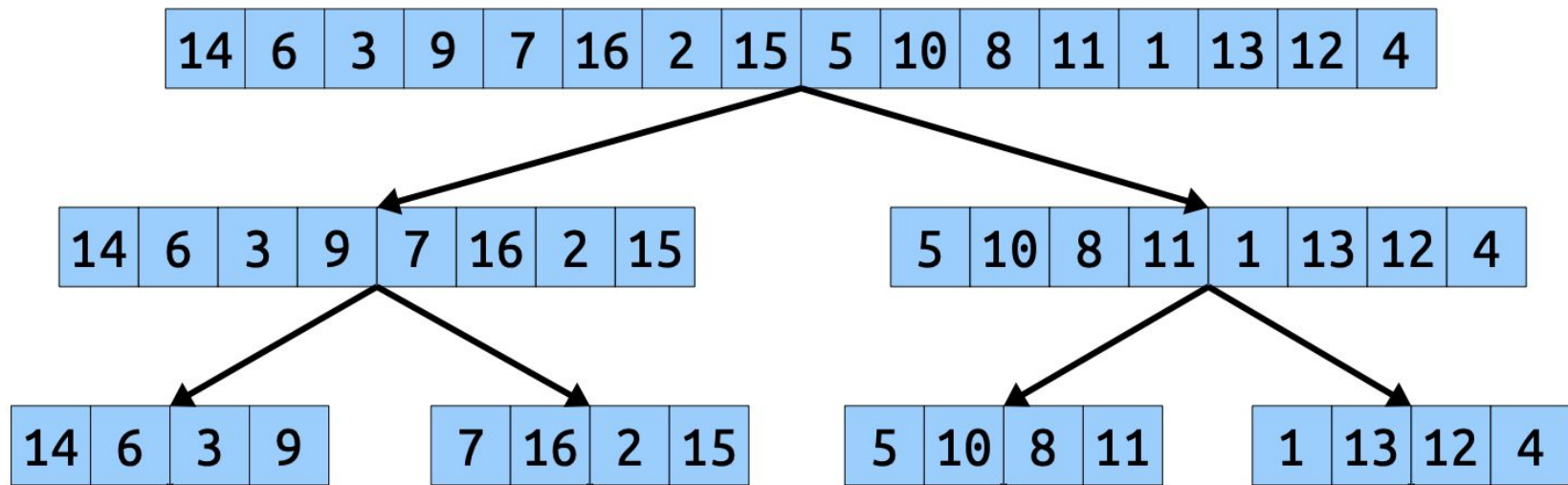
# Merge Sort

A recursive sorting algorithm!

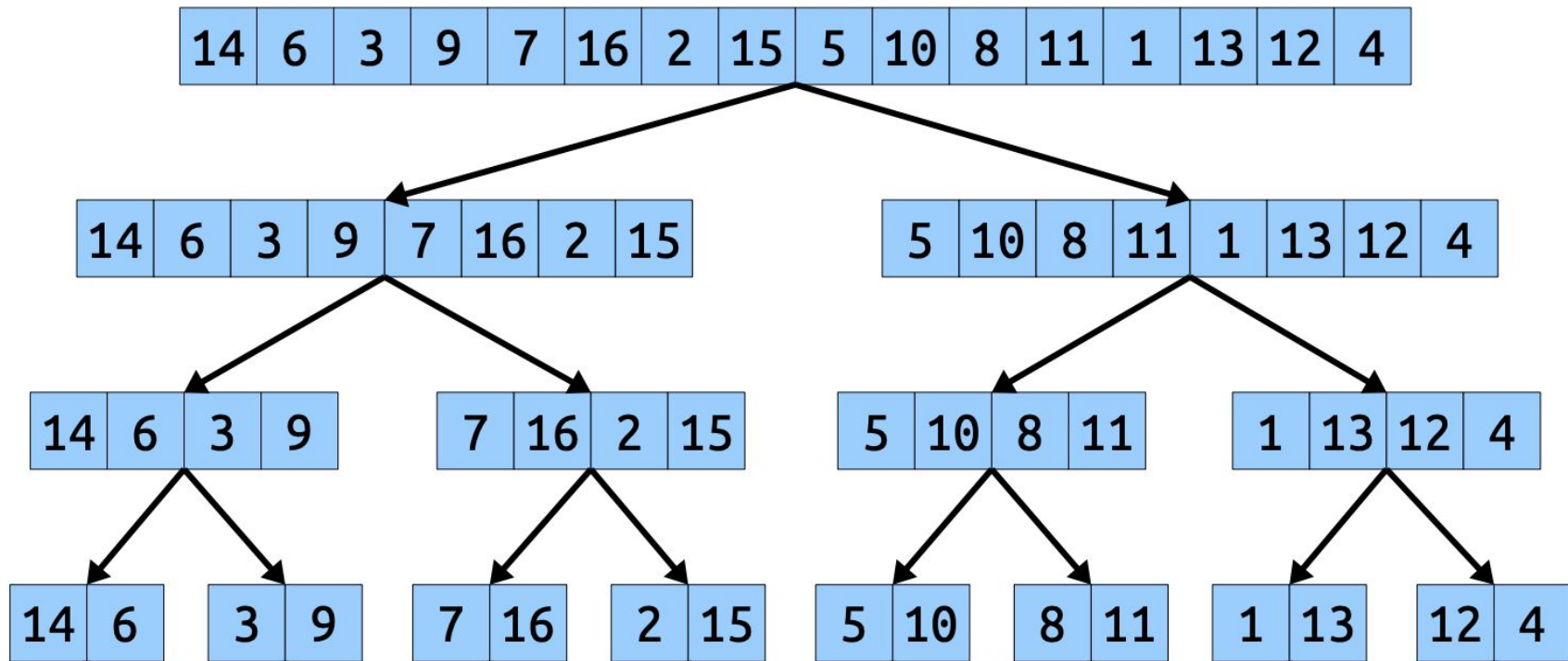
- **Base Case:**
  - An empty or single-element list is already sorted.
- **Recursive step:**
  - Break the list in half and recursively sort each part. (easy divide)
  - Use merge to combine them back into a single sorted list (hard join)

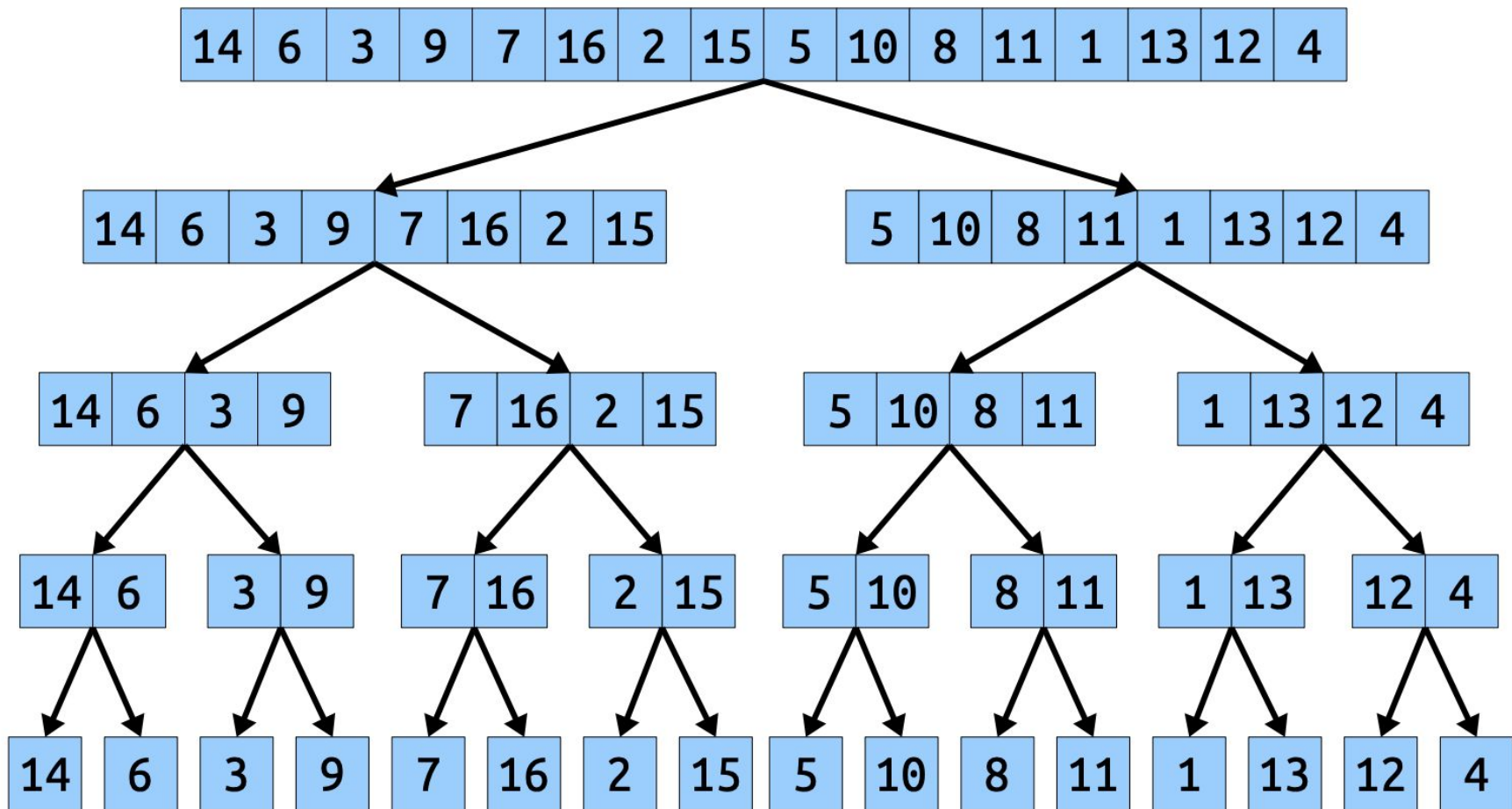
14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4
----	---	---	---	---	----	---	----	---	----	---	----	---	----	----	---





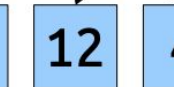






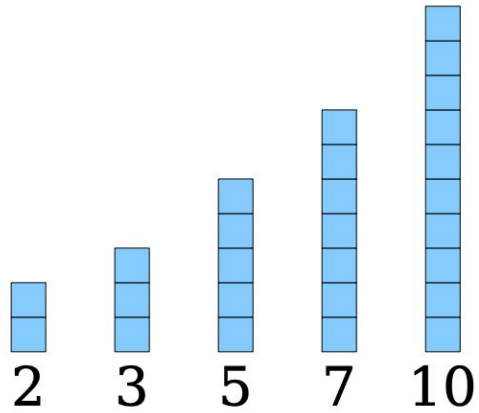


*What do we do now?  
When does the sorting  
magic happen?*

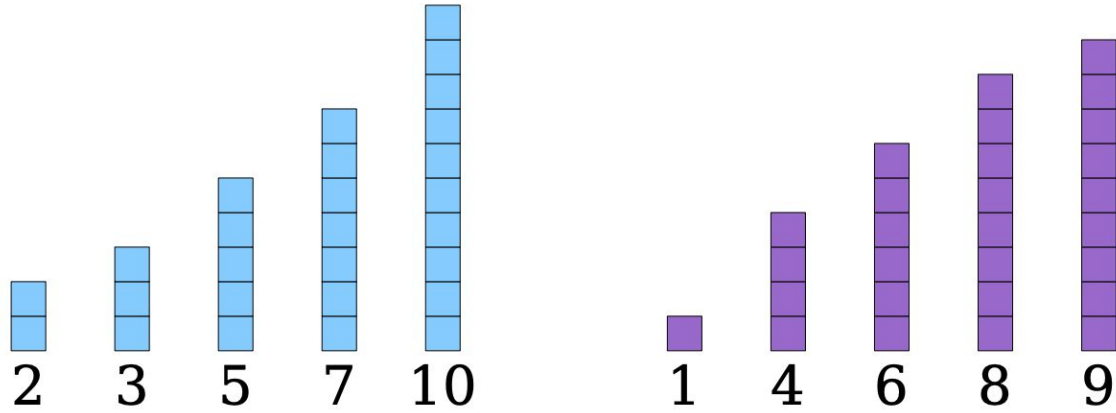


The Key Insight: Merge

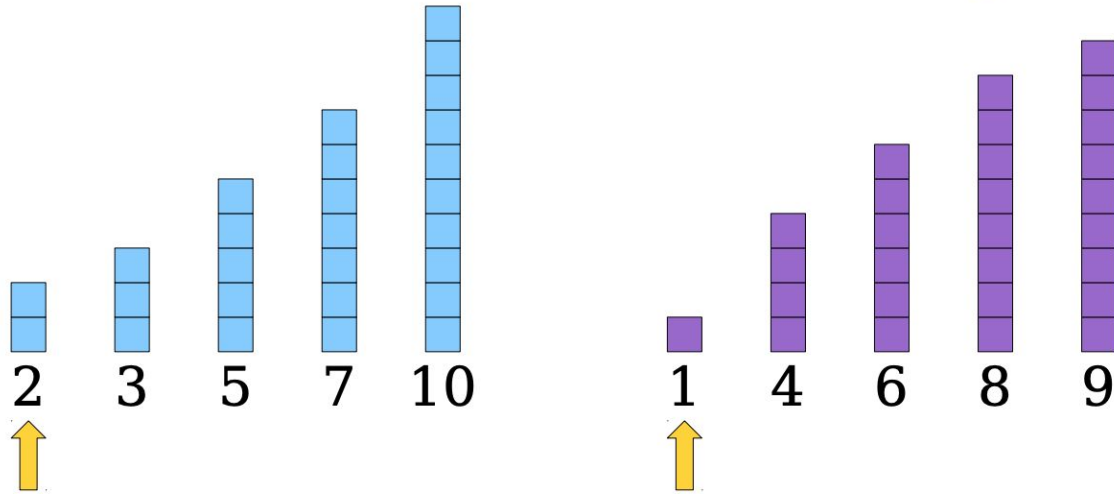
# The Key Insight: Merge



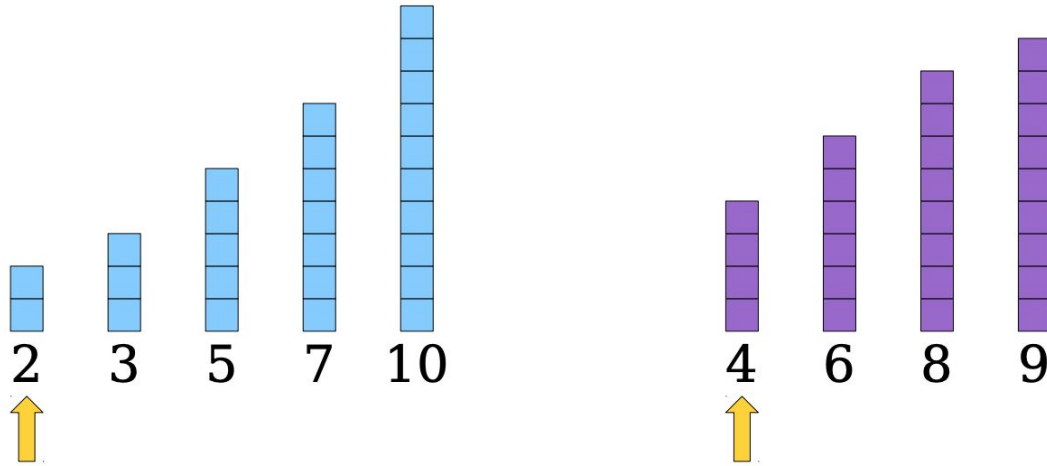
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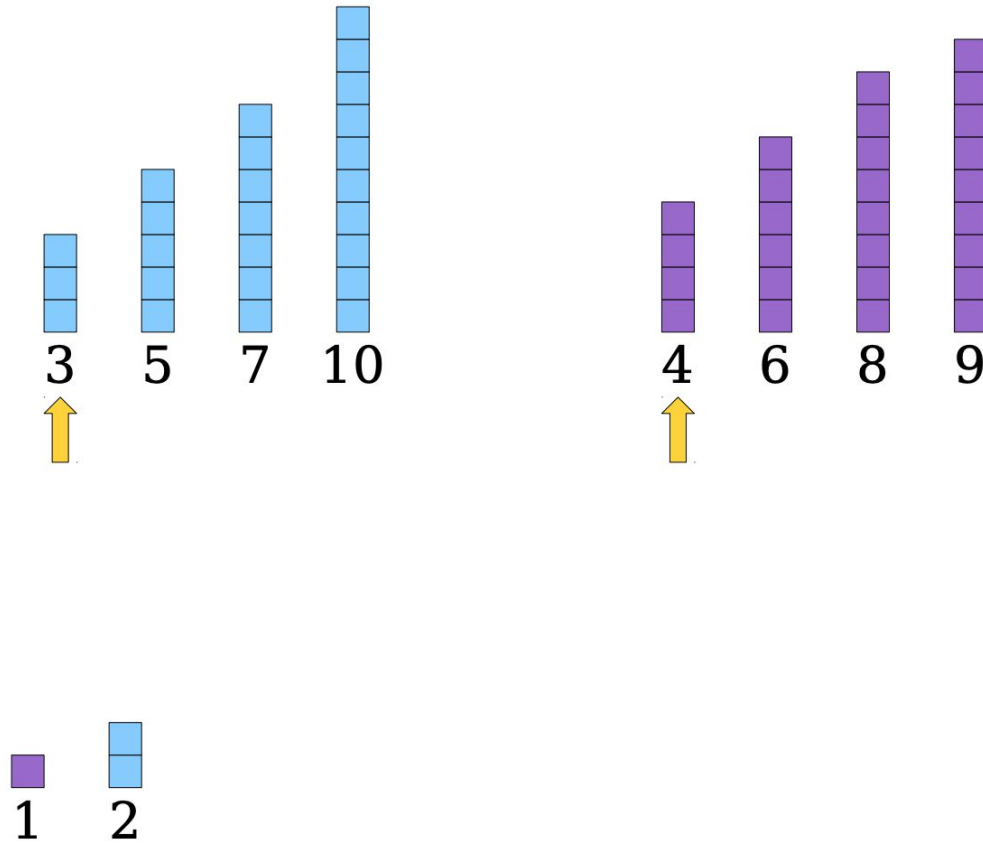
# The Key Insight: Merge



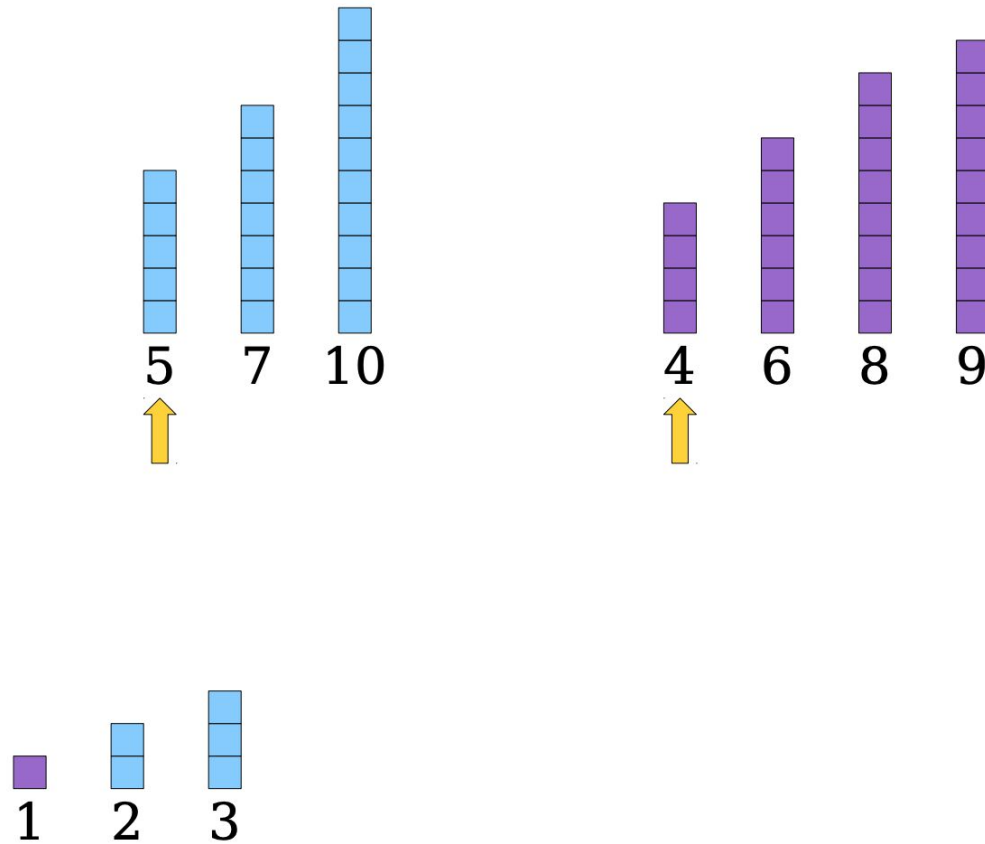
1



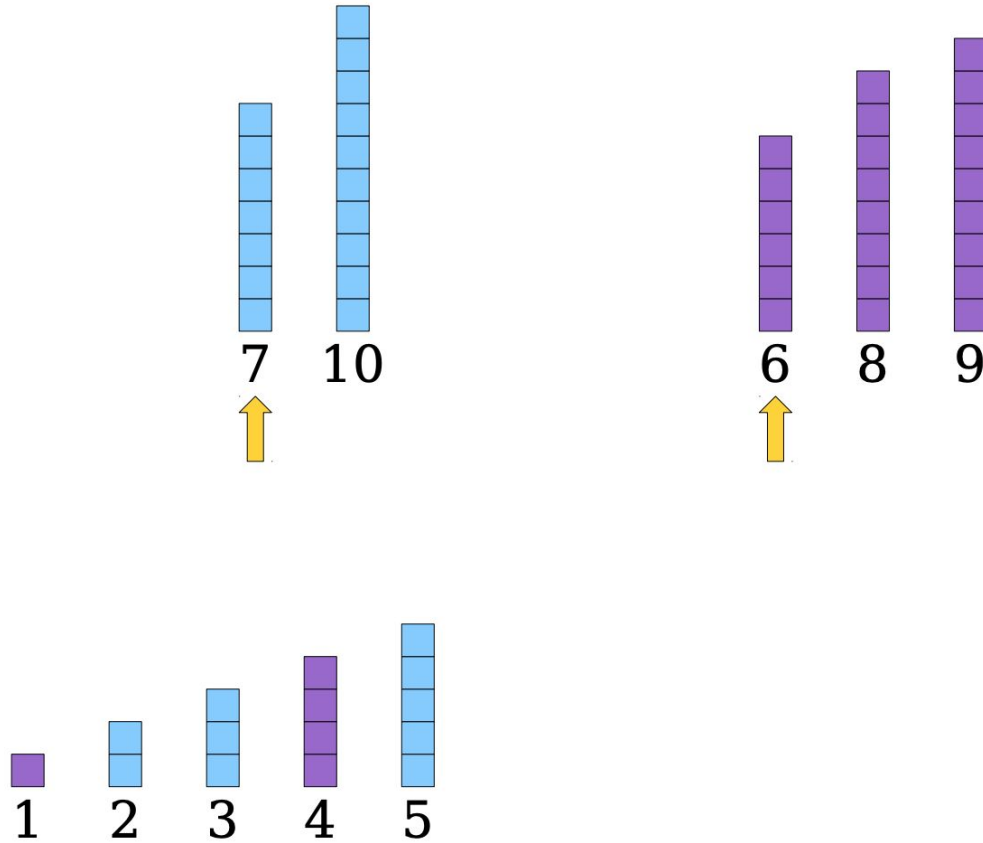
# The Key Insight: Merge



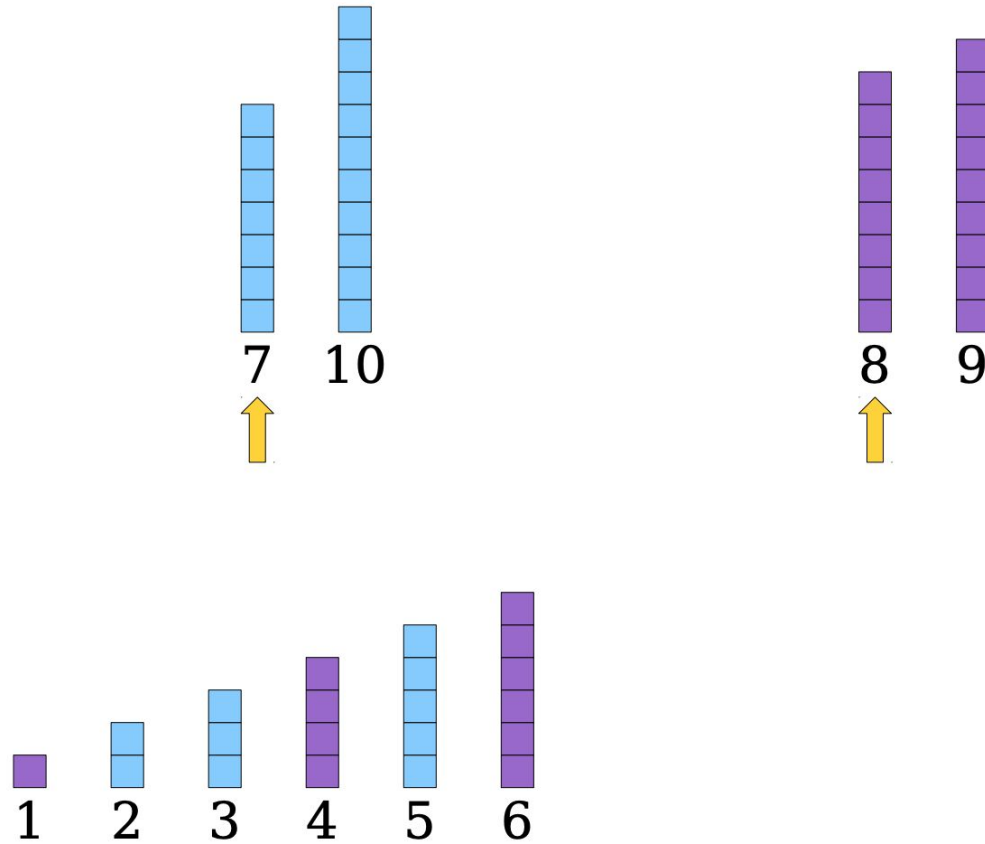
# The Key Insight: Merge



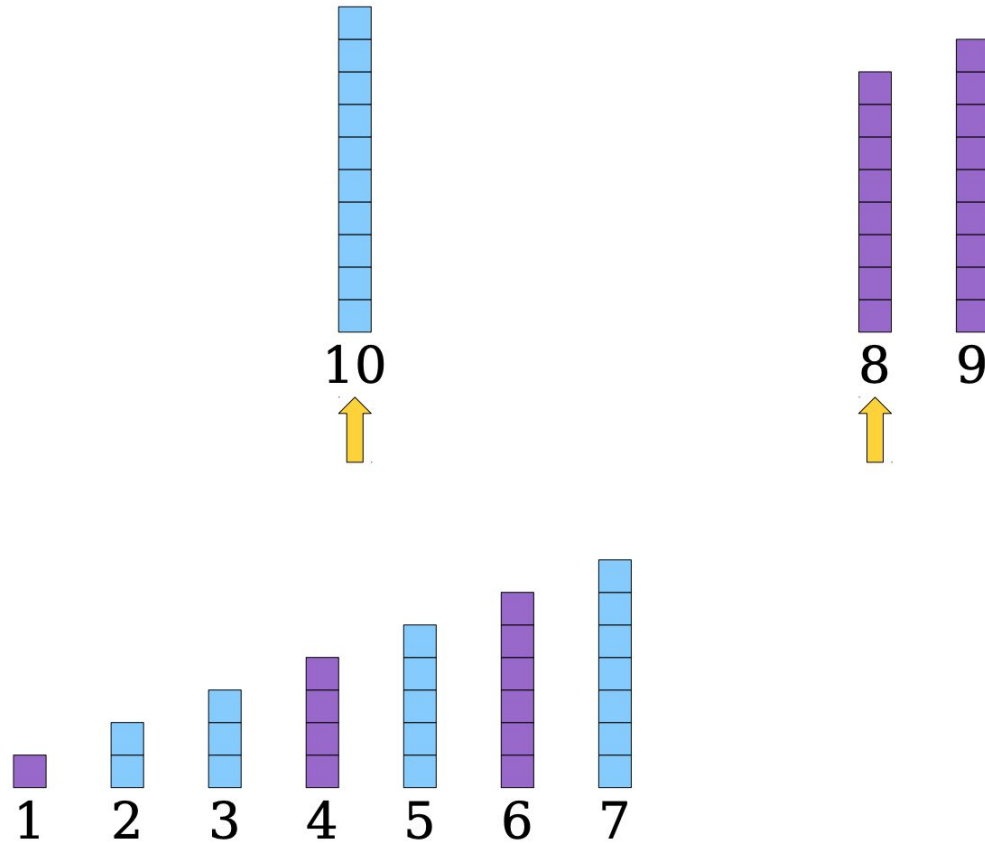
# The Key Insight: Merge



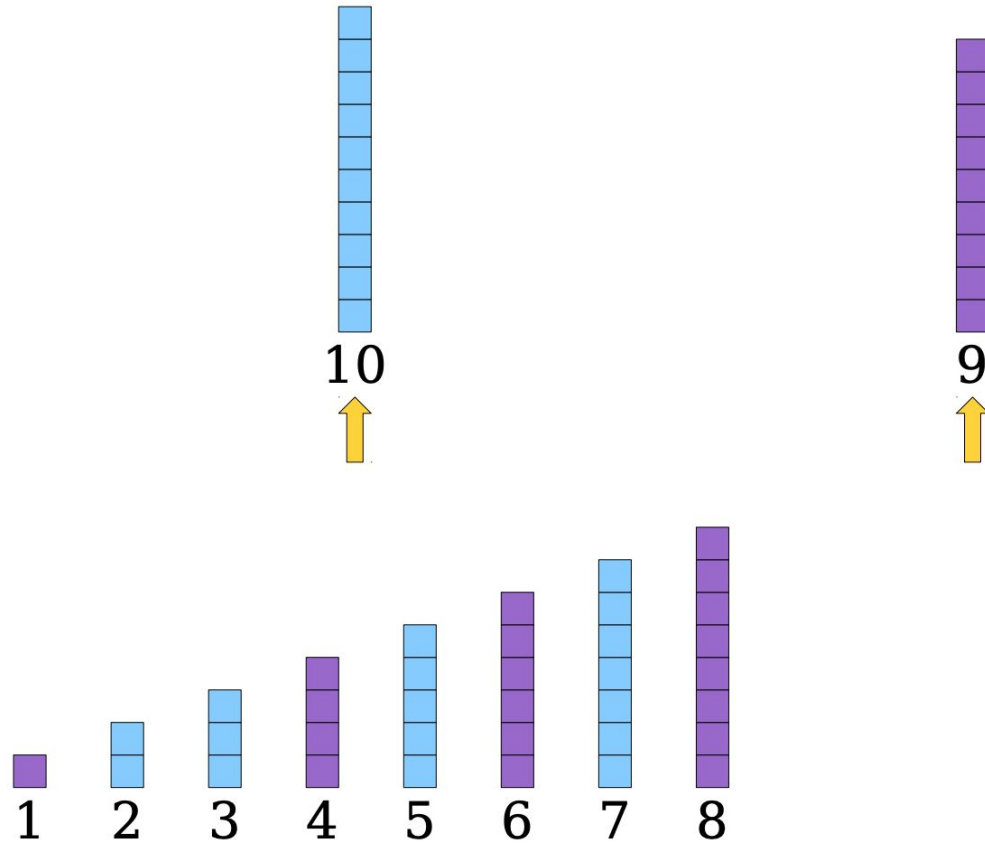
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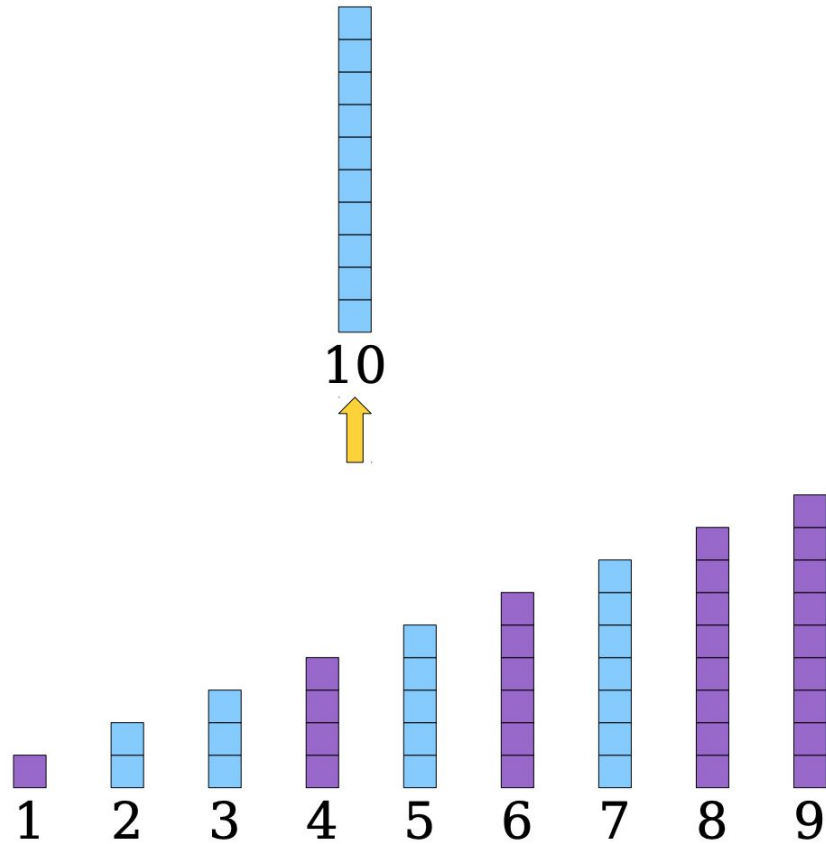
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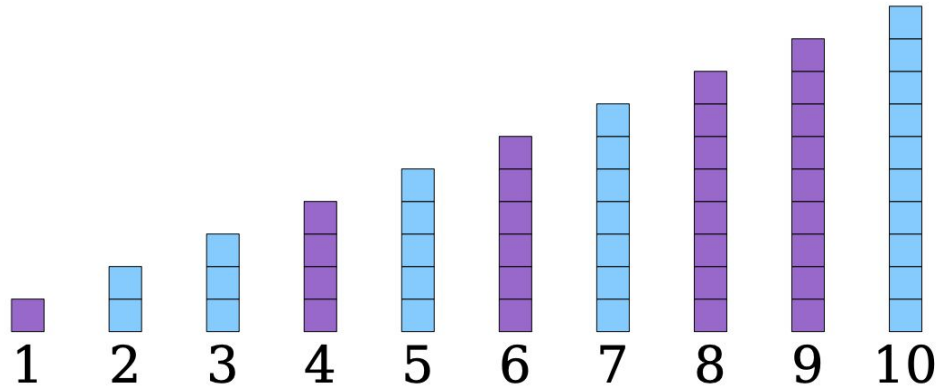
# The Key Insight: Merge



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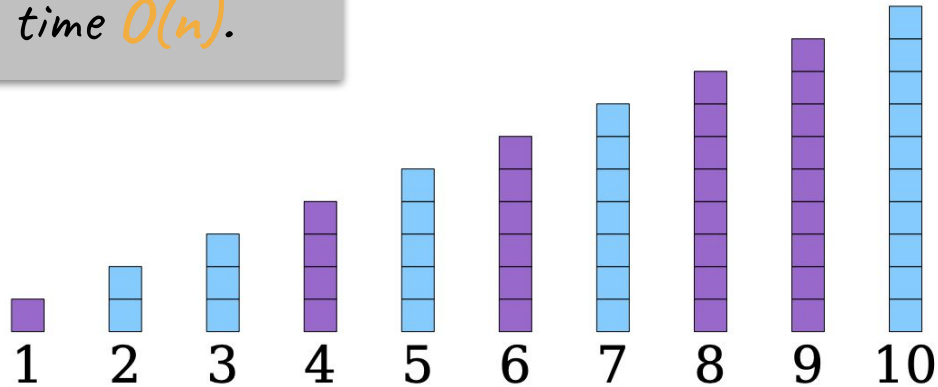
# The Key Insight: Merge





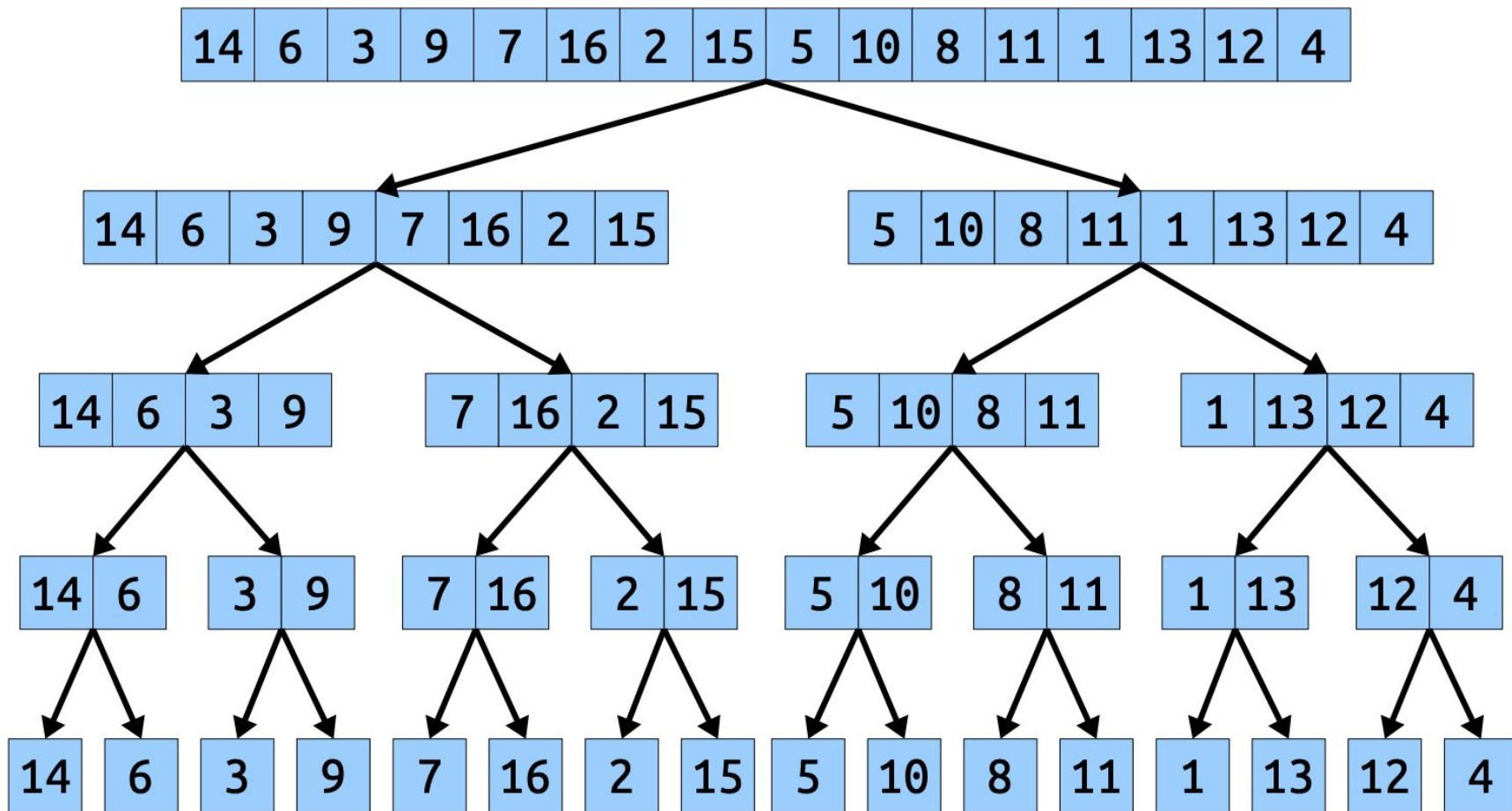
# The Key Insight: Merge

*Each step makes a single comparison and reduces the number of elements by one. If there are  $n$  total elements, this algorithm runs in time  $O(n)$ .*

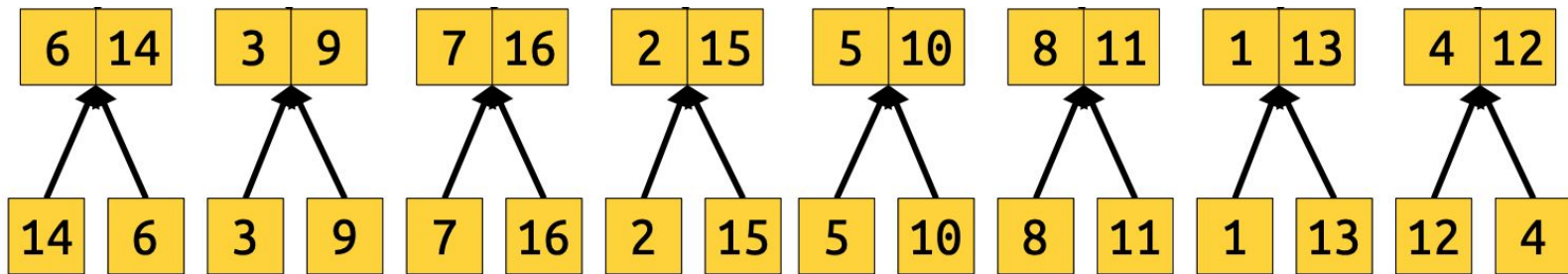


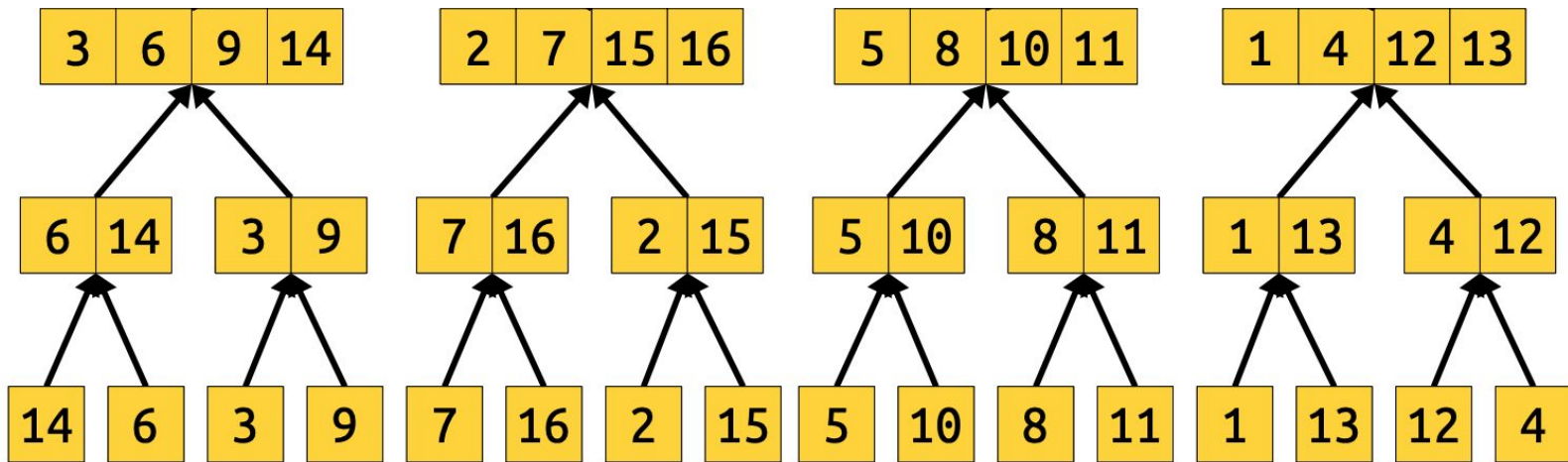
# The Key Insight: Merge

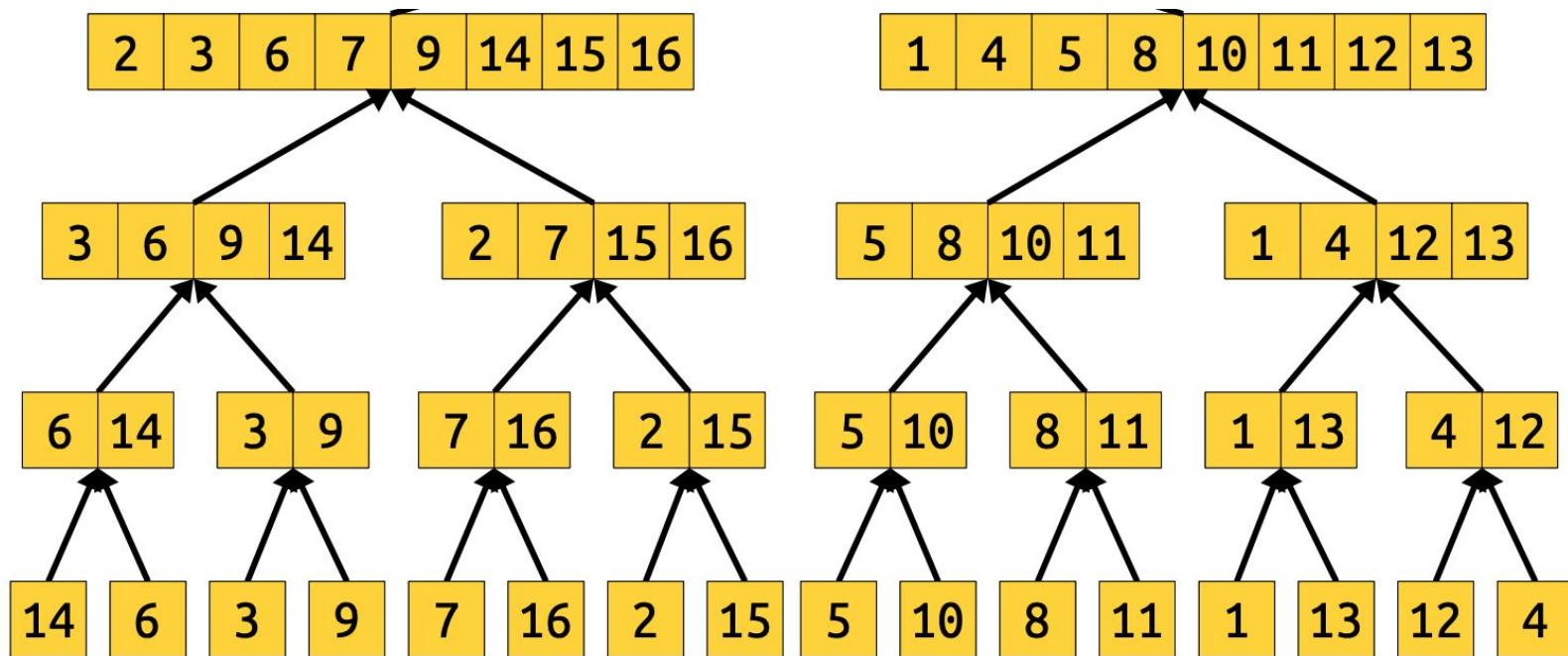
- The merge algorithm takes in two sorted lists and combines them into a single sorted list.
- While both lists are nonempty, compare their first elements. Remove the smaller element and append it to the output.
- Once one list is empty, add all elements from the other list to the output.

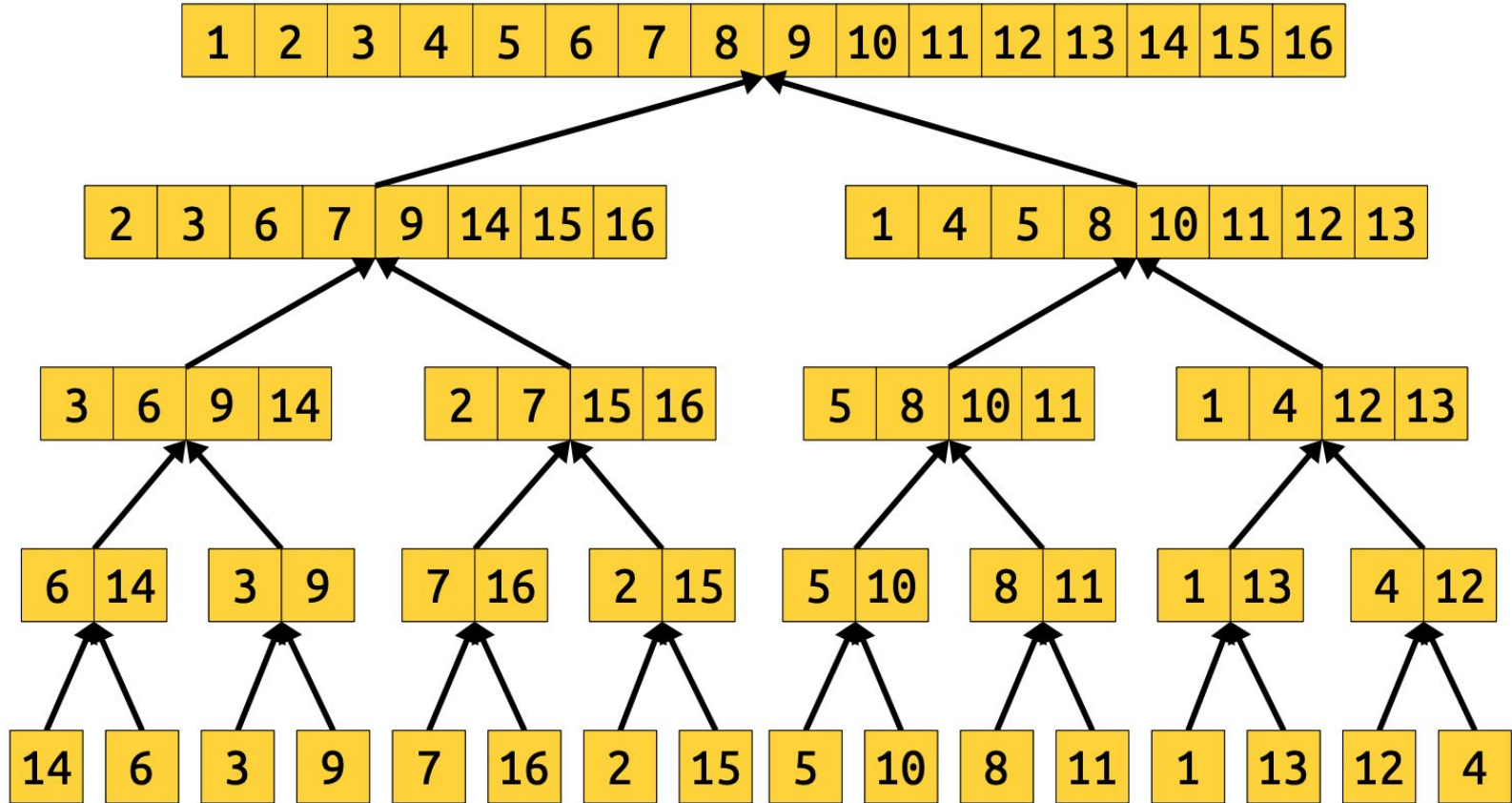


14 6 3 9 7 16 2 15 5 10 8 11 1 13 12 4

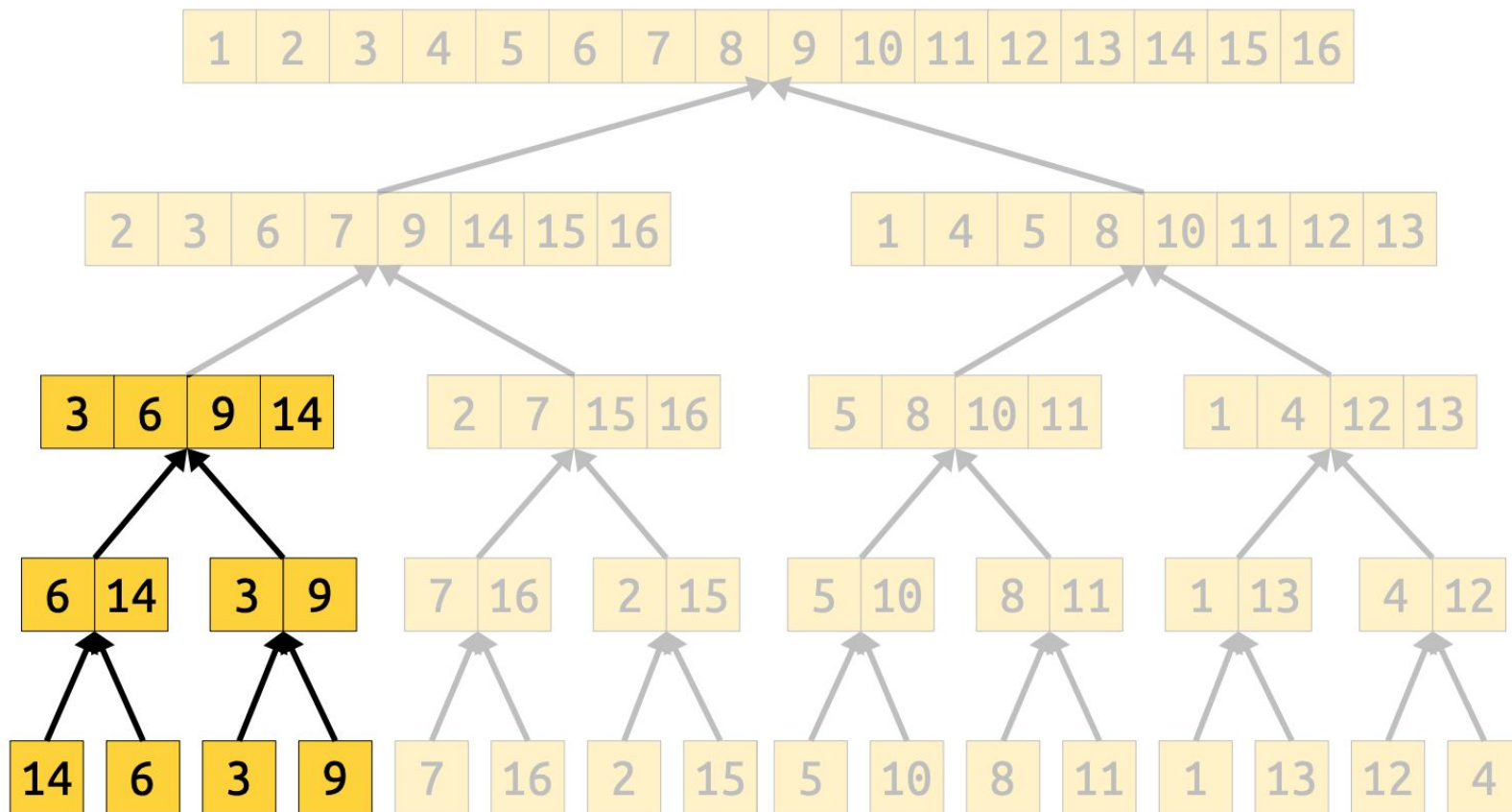












# Merge Sort

A recursive sorting algorithm!

- **Base Case:**
  - An empty or single-element list is already sorted.
- **Recursive step:**
  - Break the list in half and recursively sort each part. (easy divide)
  - Use merge to combine them back into a single sorted list (hard join)

Merge Sort – Let's look  
at the code!

```
void mergeSort(Vector<int>& vec) {  
    /* A list with 0 or 1 elements is already sorted by definition. */  
    //base case: vector is size 0 or 1, return and don't need to return a vector  
    if (vec.size() <= 1) return;  
  
    /* Split the list into two, equally sized halves */  
    Vector<int> left, right;  
    split(vec, left, right);  
  
    /* Recursively sort the two halves. */  
    mergeSort(left);  
    mergeSort(right);  
  
    /*  
    * Empty out the original vector and re-fill it with merged result  
    * of the two sorted halves.  
    */  
    vec = {}; // our merge expects an empty vector  
    merge(vec, left, right);  
}
```

## Split function definition

```
/*  
* Given a sequence of  $n$  elements (vec), splits the sequence into two  
* sublists each of size  $n/2$  and stores these sublists in the output  
* parameters left and right.  
*/  
void split(Vector<int>& vec, Vector<int>& left, Vector<int>& right){
```

# Merge function definition

```
/*  
 * Merges the sorted sequences v1 and v2 into one large, sorted  
result.  
 * Precondition: v1 and v2 are sorted, and vec is empty  
 */  
void merge(Vector<int>& vec, Vector<int>& v1, Vector<int>& v2) {
```

# Announcements

# Announcements

- Assignment 3 grades should be out! Revisions are due this **Friday at 11:59 pm**.
- Assignment 4 grace period ends **today at 11:59pm** PT.
- Assignment 5 is released! All about linked lists.
  - **YEAH Hours will be today at 5pm in Hewlett 103.**
- Lecture this Thursday will be open project work time!
  - Jenny and I will be having open OH in NVIDIA if you have questions.
  - We'll also set up areas in the lecture hall where you can discuss projects by topic so you can get feedback and ideas from your classmates.
  - Attendance is optional.
- The deadline to change your grading basis or withdraw is this Friday, July 29 at 5pm.



## Analyzing Mergesort:

How fast is this sorting algorithm?

```
void mergeSort(Vector<int>& vec) {
    /* A list with 0 or 1 elements is already sorted by definition. */
    if (vec.size() <= 1) return;

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    /* Recursively sort the two halves. */
    mergeSort(left);
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    /*
     * Empty out the original vector and re-fill it with merged result
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    vec = {};
    merge(vec, left, right);
}
```

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     * Empty out the original vector and re-fill it with merged result
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    vec = {};
    merge(vec, left, right);
}
```

} **O(n)** work

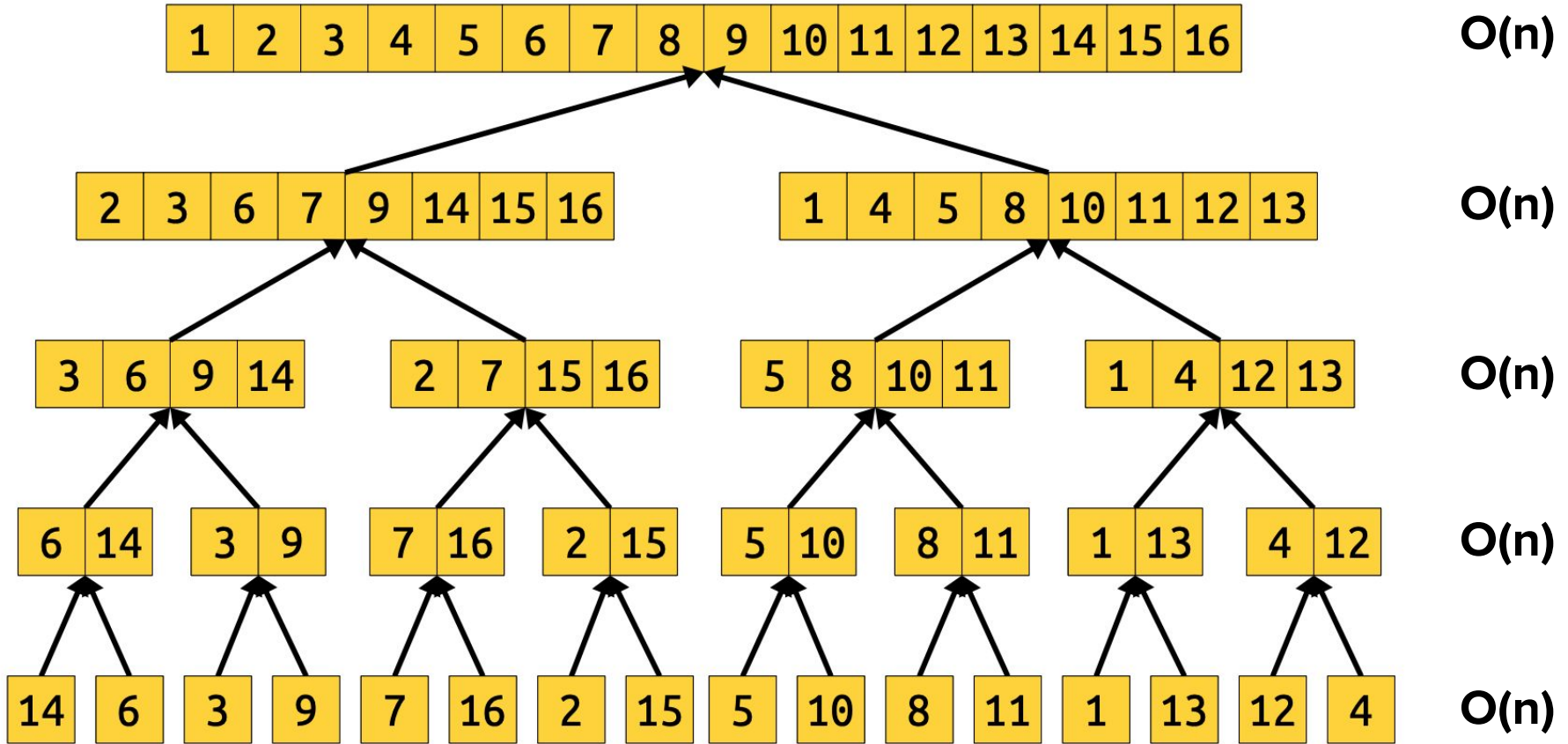
} **O(n)** work

```
void mergeSort(Vector<int>& vec) {
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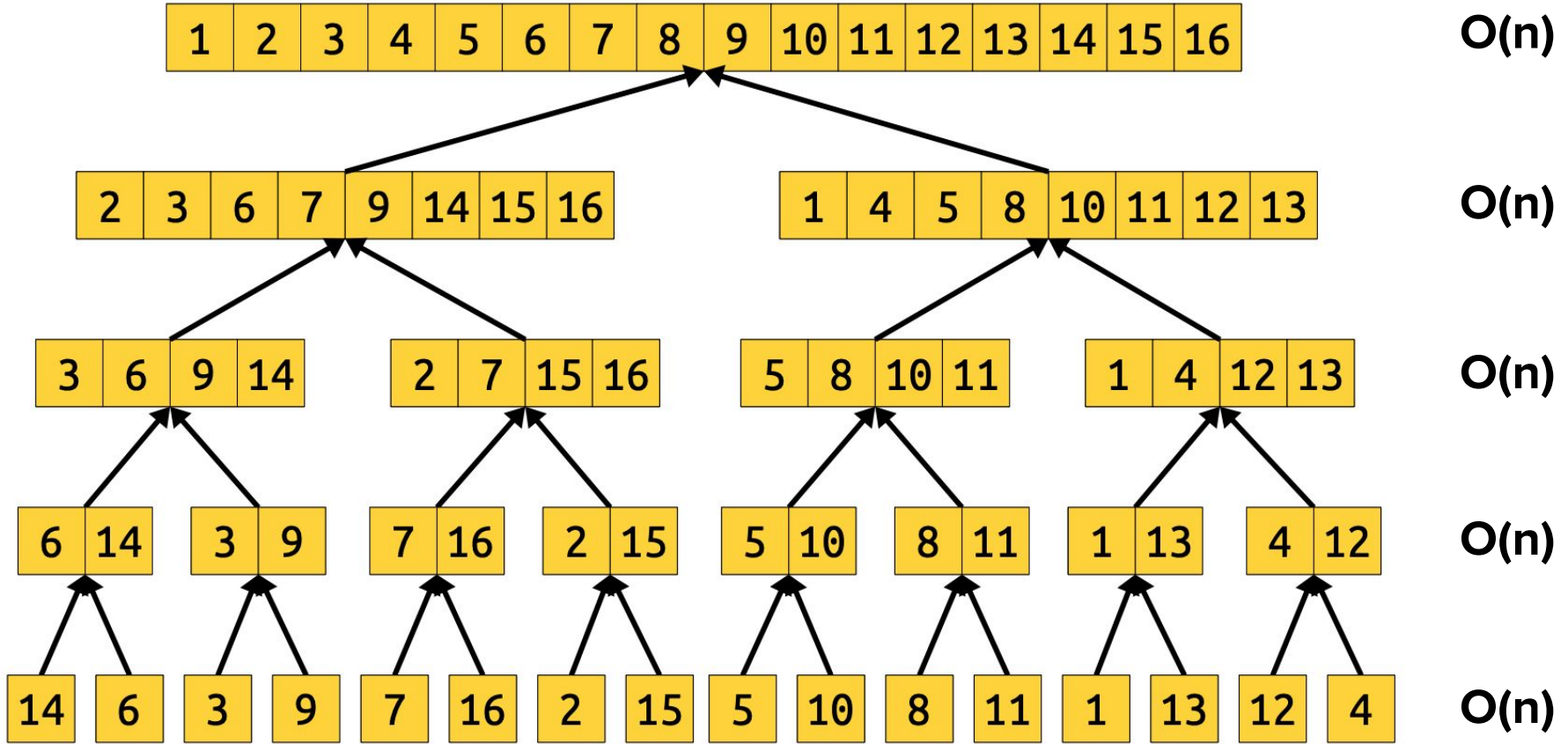
    /* Split the list into two, equally sized halves */
    Vector<int> left, right;
    split(vec, left, right);

    /* Recursively sort the two halves. */
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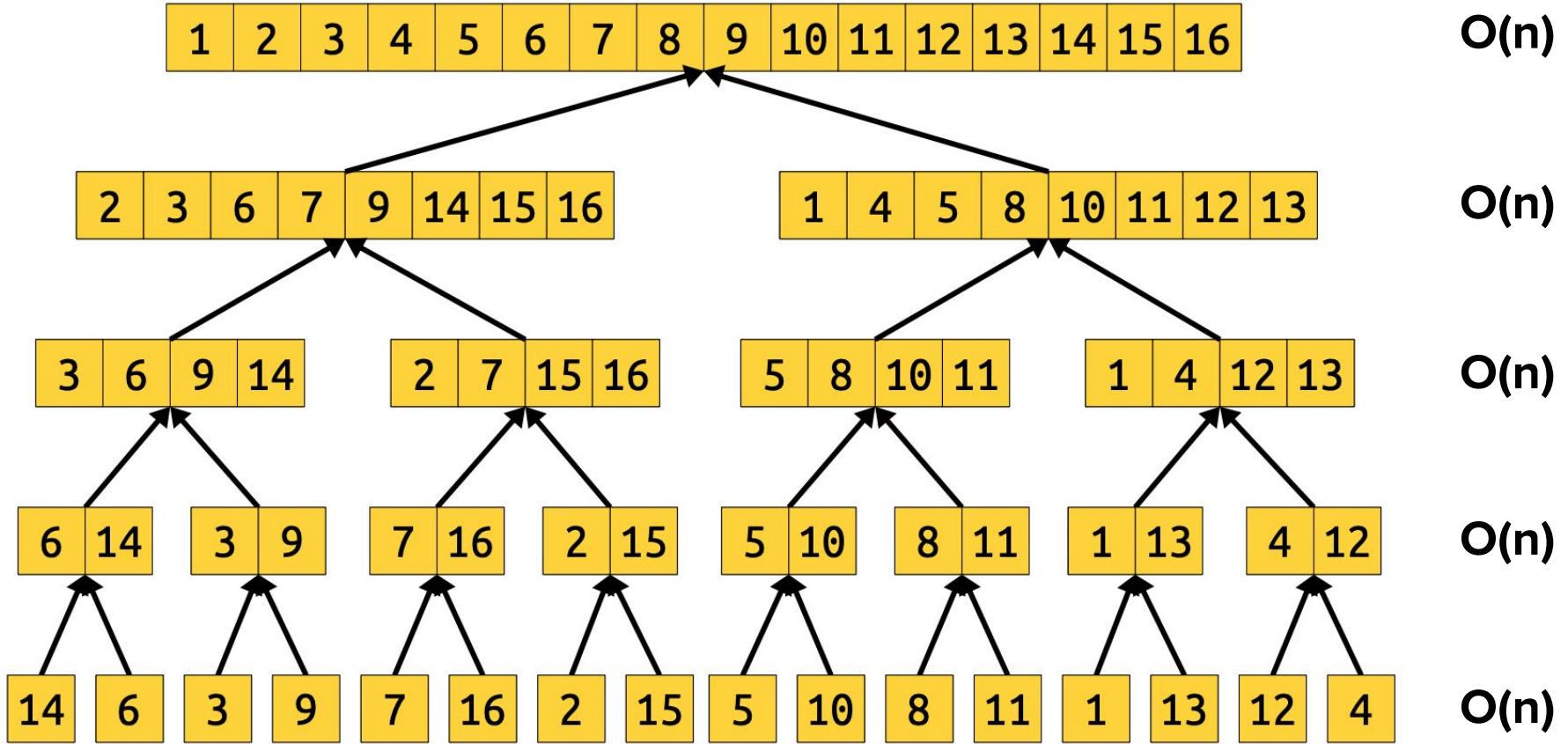
    /*
     * Empty out the original vector and re-fill it with merged result
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     */
    vec = {};
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}
```



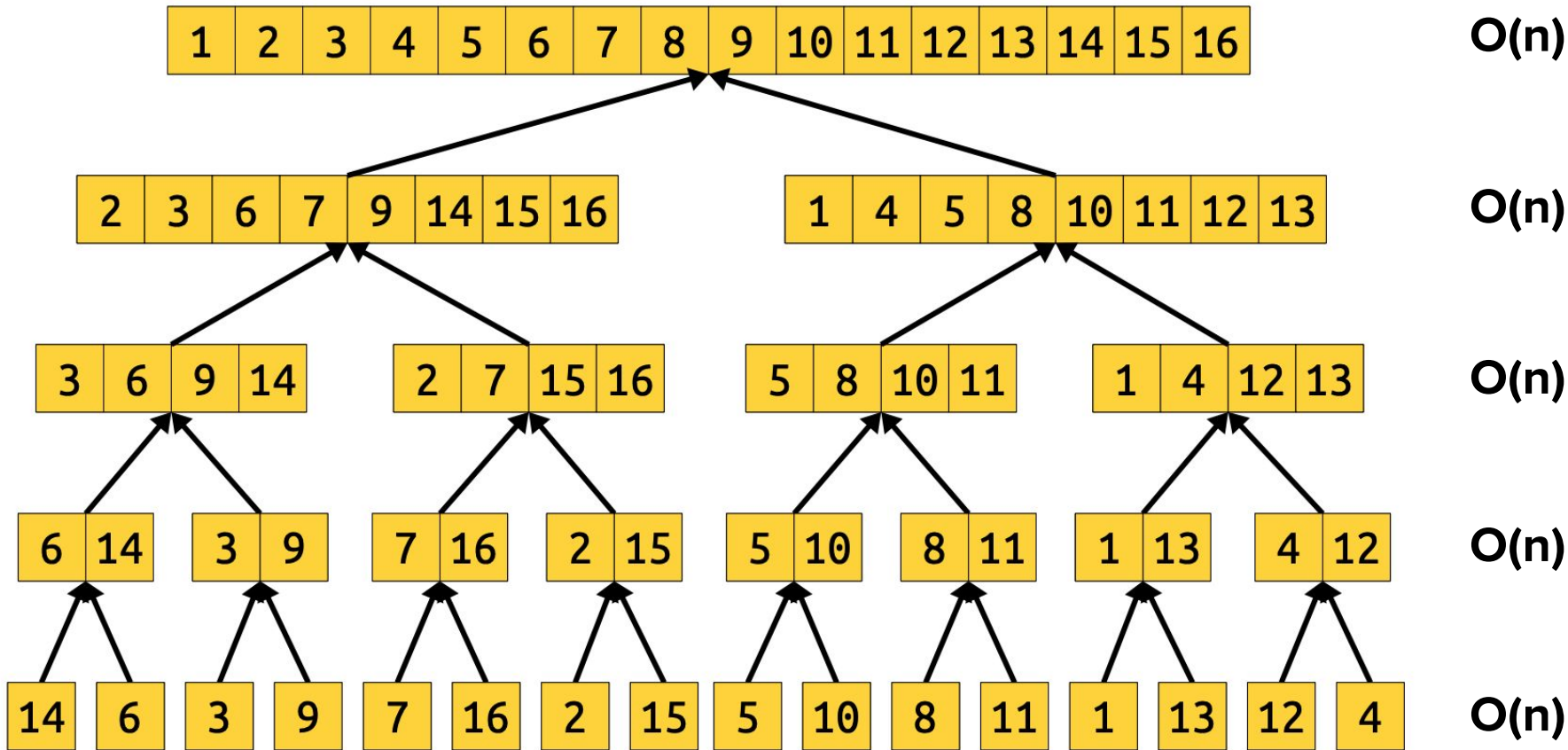
*$O(n)$  work at each level!*



*How many levels are there?*

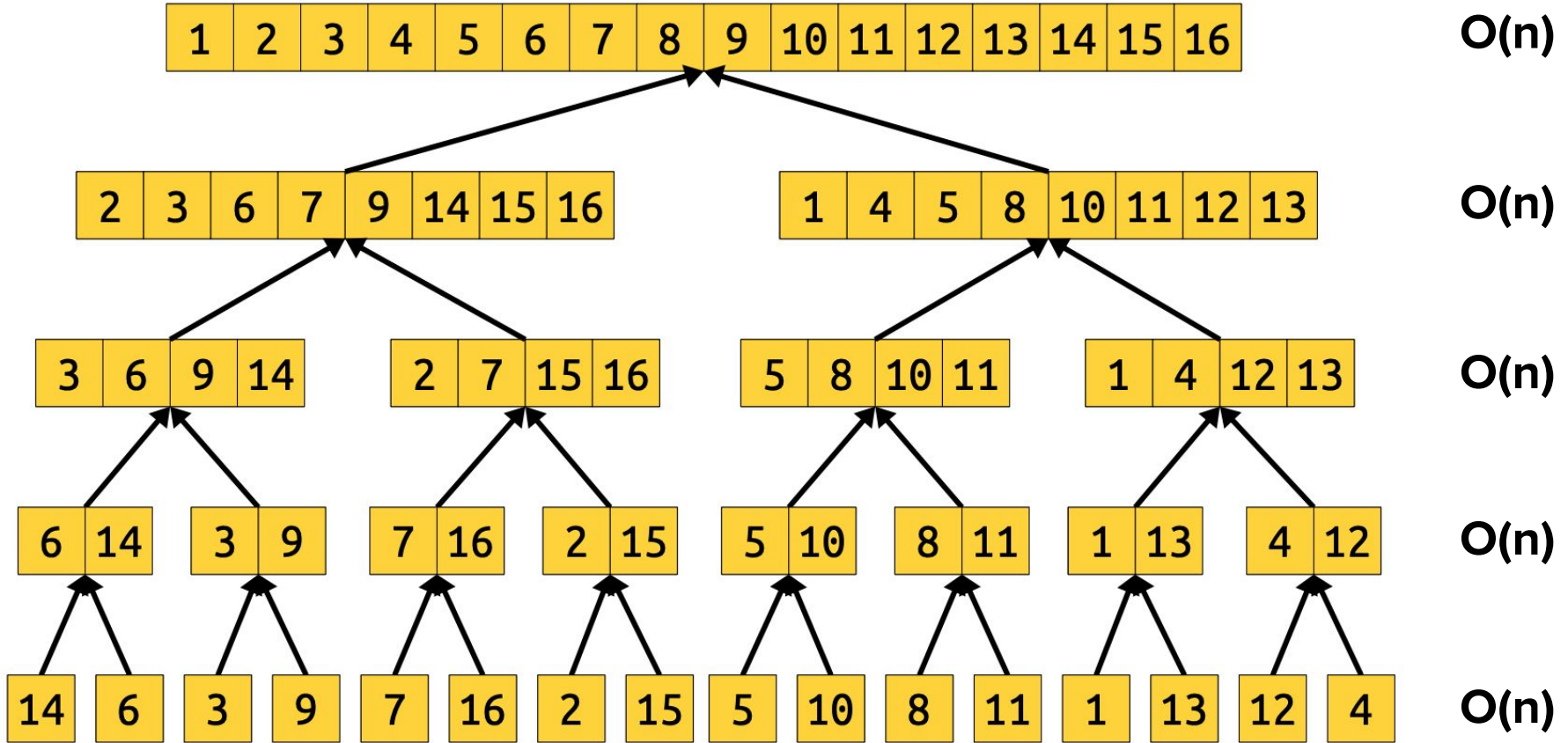


*Remember: How many times do we divide by 2?*



*$O(\log n)$  levels!*





Total work:  $O(n * \log n)$

```
void mergeSort(Vector<int>& vec) {
    /* A list with 0 or 1 elements is already sorted by definition. */
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    mergeSort(right);

    /*
     * Empty out the original vector and re-fill it with merged result
     * of the two sorted halves.
     */
    vec = {};
    merge(vec, left, right);
}
```

}  **$O(n \log n)$**  work

# Analyzing Mergesort: Can we do better?

- Mergesort runs in time  $O(n \log n)$ , which is faster than selection sort's  $O(n^2)$ .
  - Can we do better than this?
  - Let's explore one more divide-and-conquer sort!

# A Quick Historical Aside

(g) We now formulate a set of instructions to effect this 4-way decision between  $(\alpha) - (\delta)$ . We state again the contents of the short tanks already assigned:

- $\bar{1}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{2}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{3}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{4}_1) \mathcal{N}_{(1-2n)}^{m'}$   
 $\bar{5}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{6}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{7}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{8}_1) \mathcal{N}_{(1-2n)}^{m'}$   
 $\bar{9}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{10}_1) \mathcal{N}_{(1-2n)}^{m'}$      $\bar{11}_1) \dots \rightarrow \mathcal{E}$

Now let the instructions occupy the (long tank) words  $1, 2, \dots$ :

- $1.) \bar{1}_1 - \bar{5}_1$      $\sigma) \mathcal{N}_{(1-2n)}^{m' - m'}$   
 $2.) \bar{6}_1, \bar{7}_1$      $\sigma) \mathcal{N}_{(1-2n)}^{m'}$  for  $m' = n$   
 $3.) \sigma \rightarrow \bar{2}_1$      $\bar{4}_1) \mathcal{N}_{(1-2n)}^{m'}$  for  $m' \leq n$   
 $4.) \bar{1}_1 - \bar{5}_1$      $\sigma) \mathcal{N}_{(1-2n)}^{m' - m'}$   
 $5.) \bar{10}_1, \bar{8}_1$      $\sigma) \mathcal{N}_{(1-2n)}^{m'}$  for  $m' \leq n$   
 $6.) \sigma \rightarrow \bar{2}_1$      $\bar{10}_1) \mathcal{N}_{(1-2n)}^{m'}$  for  $m' \leq n$   
 $7.) \bar{2}_1 - \bar{6}_1$      $\sigma) \mathcal{N}_{(1-2n)}^{m' - m'}$   
 $8.) \bar{2}_1, \bar{6}_1$      $\sigma) \mathcal{N}_{(1-2n)}^{m'}$  for  $m' \leq n$   
                   i.e.  
 $\sigma) \mathcal{N}_{(1-2n)}^{m'}$  for  $m' = n, m' = n, m' = n, m' = n$   
                   for  $m' = n, m' = n, m' = n, m' = n$   
                   i.e. for  $(\alpha), (\beta), (\gamma), (\delta)$ , respectively.  
 $9.) \sigma \rightarrow \bar{11}_1$      $\bar{11}_1) 1_1, 2_1, 3_1 \rightarrow \mathcal{E}$  for  $(\alpha), (\beta), (\gamma), (\delta)$ , respectively.  
 $10.) \bar{11}_1 \rightarrow \mathcal{E}$

~~End of the set of instructions for the 4-way decision~~

Now  
 $\bar{11}_1) 1_1, 2_1, 3_1 \rightarrow \mathcal{E}$  for  $(\alpha), (\beta), (\gamma), (\delta)$ , respectively.

Thus at the end of this phase  $\mathcal{E}$  is at  $1_1, 2_1, 3_1, 4_1$ , according to which case  $(\alpha), (\beta), (\gamma), (\delta)$  holds.

(h) We now pass to the case  $(\alpha)$ . This has ~~two~~ 2 subcases  $(\alpha_1)$  and  $(\alpha_2)$ , according to whether  $x_1 \geq x_2$  or  $x_1 < x_2$ . According to which of the 2 subcases holds,  $\mathcal{E}$  must be sent to the place where its instructions begin, say the (long tank) words  $1_{01}, 1_{02}$ . Their numbers must

~~be the same as the numbers of the instructions in the short tanks~~

FIG. 1. The original manuscript.

# A Quick Historical Aside

- Mergesort was one of the first algorithms developed for computers as we know them today.
- It was invented by John von Neumann in 1945 (!) as a way of validating the design of the first “modern” (stored-program) computer.
- Want to learn more about what he did? Check out [this article](#) by Stanford’s very own Donald Knuth.

# Quicksort

# Quicksort Algorithm

1. **Partition** the elements into three categories based on a chosen **pivot** element:
  - Elements **smaller** than the pivot
  - Elements **equal** to the pivot
  - Elements **larger** than the pivot

# Quicksort Algorithm

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 Our *divide* step (hard divide)!



# Quicksort Algorithm

1. **Partition** the elements into three categories based on a chosen **pivot** element:
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  - Elements **larger** than the pivot
2. **Recursively sort** the two partitions that are not equal to the pivot (smaller and larger elements).
  - Now our smaller elements are in sorted order, and our larger elements are also in sorted order!

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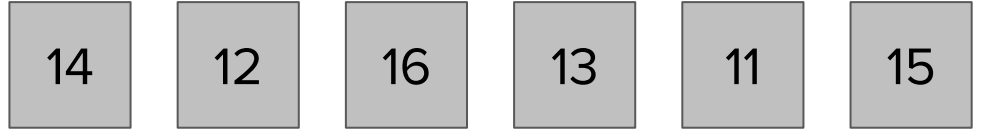
# Quicksort Algorithm

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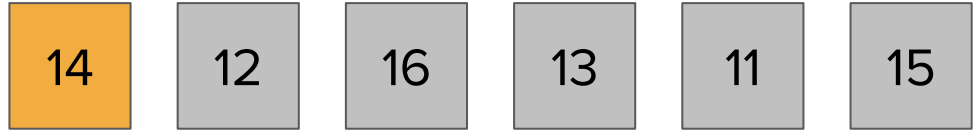


*Our **join** step!  
(easy join)*

Input of unsorted elements:

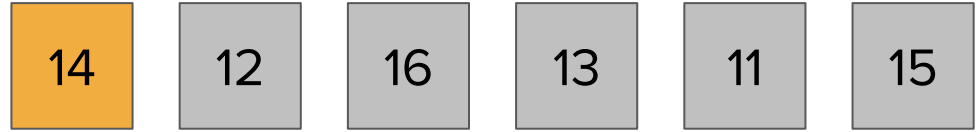


Input of unsorted elements:



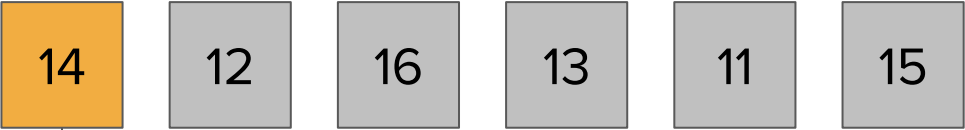
*Choose the first element as the pivot.*

Input of unsorted elements:



*Partition elements into smaller than, equal to, and greater than the pivot.*

Input of unsorted elements:



Input of unsorted elements:

14

12

16

13

11

15

12

13

11

16

15

*Recursively sort the smaller partition  
for pivot 14!*



Input of unsorted elements:

14

12

16

13

11

15

12

13

11

16

15

*Choose the first element as the pivot.*

Input of unsorted elements:

14

12

16

13

11

15

12

13

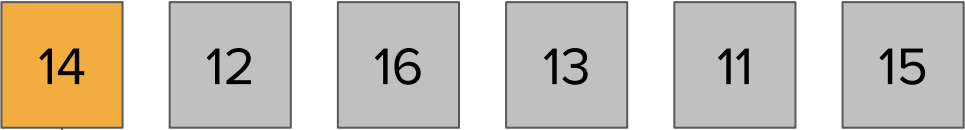
11

16

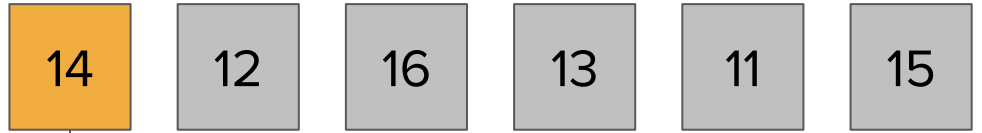
15

*Partition elements into smaller than,  
equal to, and greater than the pivot.*

Input of unsorted elements:

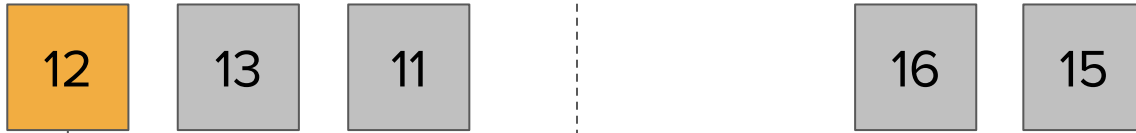
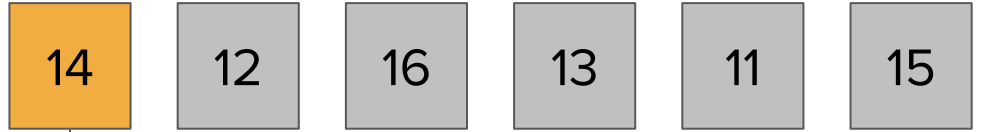


Input of unsorted elements:



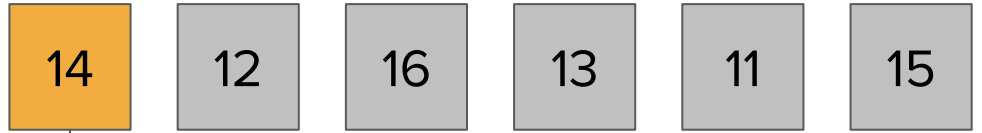
*Recursively sort the smaller partition for pivot 12!*

Input of unsorted elements:



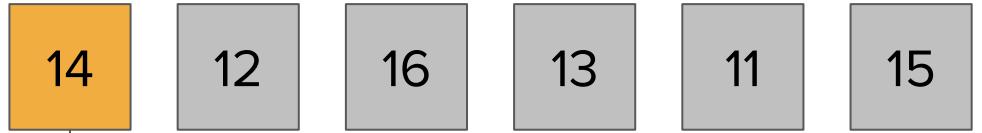
*Only one element so we're done!*

Input of unsorted elements:



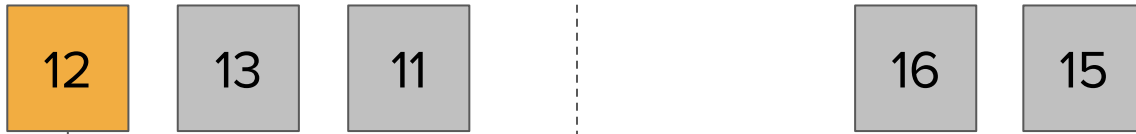
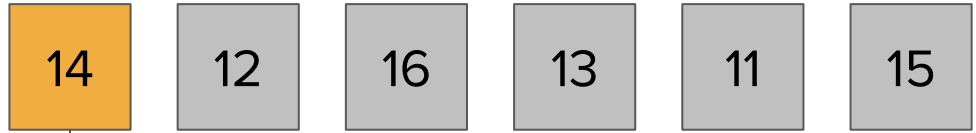
*Recursively sort the larger  
partition for pivot 12!*

Input of unsorted elements:



*Only one element so we're done!*

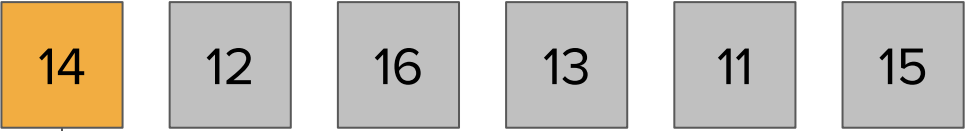
Input of unsorted elements:



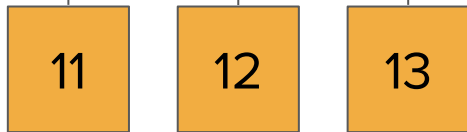
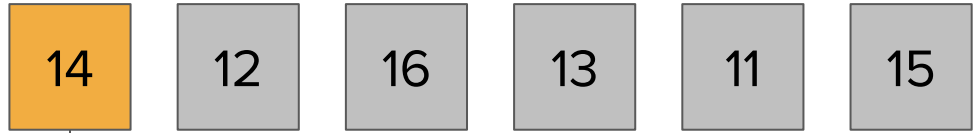
*Now we can concatenate smaller than, equal to, and greater than for the pivot 12.*



Input of unsorted elements:

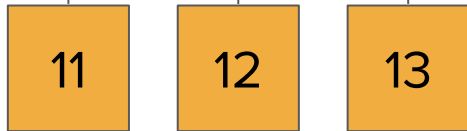
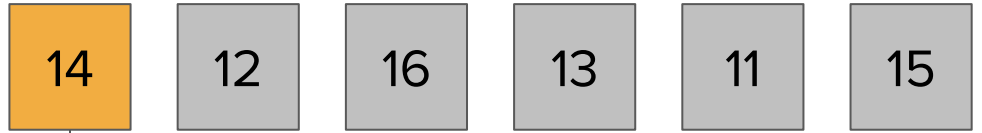


Input of unsorted elements:



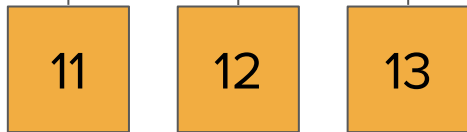
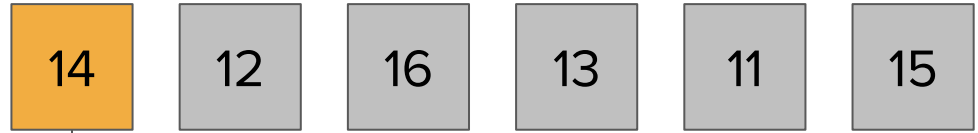
*Recursively sort the larger partition for pivot 14!*

Input of unsorted elements:



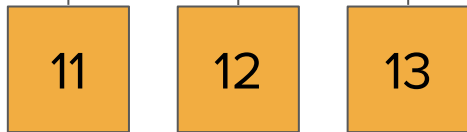
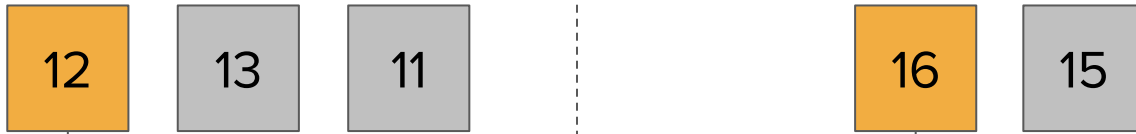
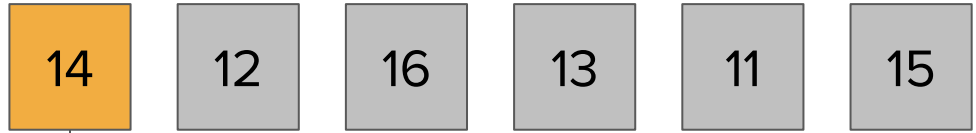
*Choose the first element as the pivot.*

Input of unsorted elements:



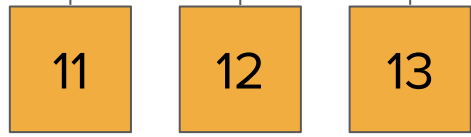
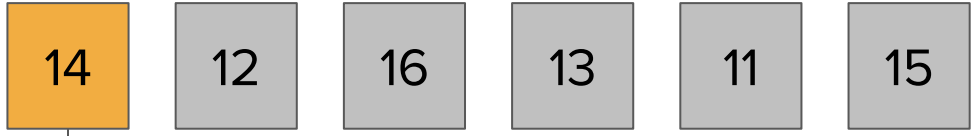
*Partition elements into smaller than, equal to, and greater than the pivot.*

Input of unsorted elements:



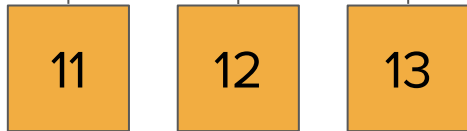
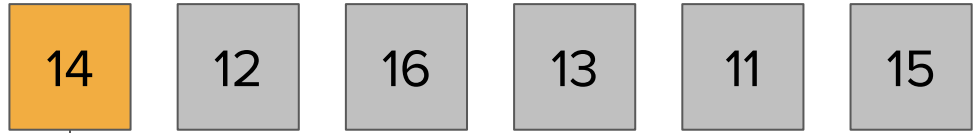
*Recursively sort the smaller partition for pivot 16!*

Input of unsorted elements:



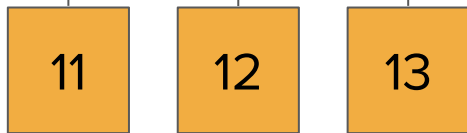
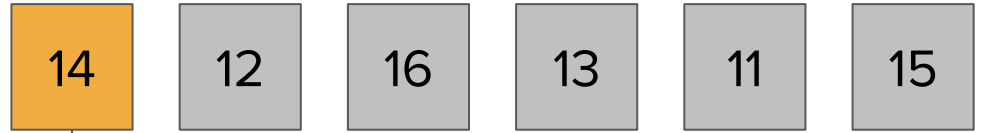
*Only one element so we're done!*

Input of unsorted elements:



*Recursively sort the larger partition for pivot 16!*

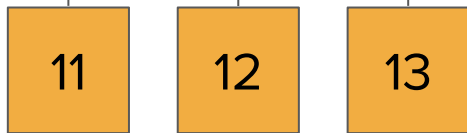
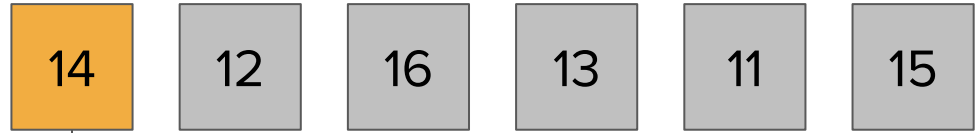
Input of unsorted elements:



*No elements in that partition so we're done!*



Input of unsorted elements:

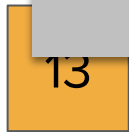
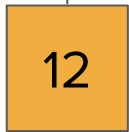
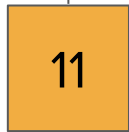
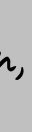
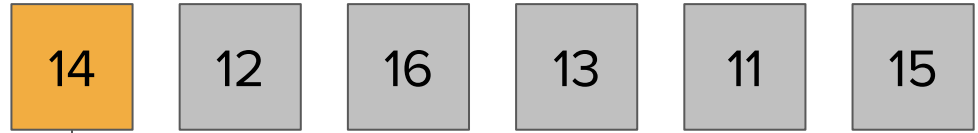


*Now we can concatenate smaller than, equal to, and greater than for the pivot 16.*

Input of unsorted elements:

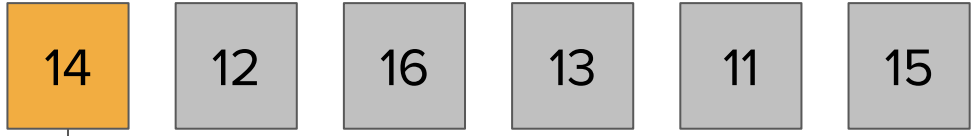


Input of unsorted elements:

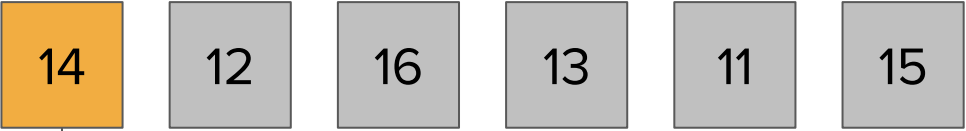


*Now we can concatenate smaller than, equal to, and greater than for the pivot 14. (the original pivot!)*

Input of unsorted elements:



Input of unsorted elements:



Sorted!

# Quicksort Algorithm

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2. **Recursively sort** the two partitions that are not equal to the pivot (smaller and larger elements).
  - Now our smaller elements are in sorted order, and our larger elements are also in sorted order!
3. **Concatenate** the three now-sorted partitions together.

# Quicksort Pseudocode

```
void quickSort(Vector<int>& vec){
    /* A list with 0 or 1 elements is already sorted by definition. */
    if vector length is <= 1: return

    /* Pick the pivot and partition the list into three components.
     * 1) elements less than the pivot
     * 2) elements equal to the pivot
     * 3) elements greater than the pivot
     */
    Define three empty lists: less, equal, greater
    pivot = first element of vector (arbitrary choice)
    partition(vec, less, equal, greater, pivot)

    /* Recursively sort the two unsorted components. */
    quickSort(less)
    quickSort(greater)

    /* Concatenate newly sorted results and store in original vector */
    concatenate(vec, less, equal, greater)
}
```

# Quicksort Takeaways

- Our “divide” step = partitioning elements based on a pivot
- Our recursive call comes in between dividing and joining
  - Base case: One element or no elements to sort!
- Our “join” step = combining the sorted partitions
- Unlike in merge sort where most of the sorting work happens in the “join” step, our sorting work occurs primarily at the “divide” step for quicksort (when we sort elements into partitions).



# Quicksort Efficiency Analysis

- Similar to Merge Sort, Quicksort also has  $O(N \log N)$  runtime in the average case.
  - With good choice of pivot, we split the initial list into roughly two equally-sized parts every time.
  - Thus, we reach a depth of about  $\log N$  split operations before reaching the base case.
  - At each level, we do  $O(n)$  work to partition and concatenate.

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- However, Quicksort performance can degrade to  $O(N^2)$  with poor choice of pivot!
  - Come talk to us after class if you're interested in why!

# The Limit Does Exist

- There is a **fundamental limit** on the efficiency of comparison-based sorting algorithms.

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- However, Quicksort performance can degrade to  $O(N^2)$  with poor choice of pivot!
  - Come talk to us after class if you're interested in why!
- The ultimate question: Can we do better?
  - From a space efficiency perspective, yes, there are versions of Quicksort that don't require making many copies of the list (in-place Quicksort). But from a runtime efficiency perspective...

# The Limit Does Exist

- There is a **fundamental limit** on the efficiency of comparison-based sorting algorithms.
- You can prove that it is not possible to guarantee a list has been sorted unless you have done **at minimum  $O(N \log N)$  comparisons**.
  - Take CS161 to learn how to write this proof!

# The Limit Does Exist

- There is a **fundamental limit** on the efficiency of comparison-based sorting algorithms.
- You can prove that it is not possible to guarantee a list has been sorted unless you have done **at minimum  $O(N \log N)$  comparisons**.
  - Take CS161 to learn how to write this proof!
- Thus, we can't do better (in Big-O terms at least) than Merge Sort and Quicksort!
  - Take CS161 to learn about how there are actually clever non-comparison-based sorting algorithms that are able to break this limit.

# Final Advice

# Assignment 6 Tips

- When implementing the sorting algorithm on linked lists, it is strongly recommended to implement helper functions for the divide/join components of the algorithm.
  - For quicksort this means having helper functions for the partition and concatenate operations



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- These helper functions should be implemented iteratively, but the overall sorting algorithm itself operates recursively. Mind the distinction!

# Assignment 6 Tips

- When implementing the sorting algorithm on linked lists, it is strongly recommended to implement helper functions for the divide/join components of the algorithm.
  - For quicksort this means having helper functions for the partition and concatenate operations
- These helper functions should be implemented iteratively, but the overall sorting algorithm itself operates recursively. Mind the distinction!
- Write tests for your helper functions first! Then, write end-to-end tests for your sorting function.

# Summary

<https://www.toptal.com/developers/sorting-algorithms>

 Play All	 Insertion	 Selection	 Bubble	 Shell	 Merge	 Heap	 Quick	 Quick3
 Random								
 Nearly Sorted								
 Reversed								
 Few Unique								

## Sorting Big-O Cheat Sheet

Sort	Worst Case	Best Case	Average Case
Insertion	$O(n^2)$	$O(n)$	$O(n^2)$
Selection	$O(n^2)$	$O(n^2)$	$O(n^2)$
Merge	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$
Quicksort	$O(n^2)$	$O(n \log n)$	$O(n \log n)$

# Sorting

- Sorting is a powerful tool for organizing data in a meaningful format!
- There are many different methods for sorting data:
  - Selection Sort
  - Insertion Sort
  - Mergesort
  - Quicksort
  - And many more...
- Understanding the different runtimes and tradeoffs of the different algorithms is important when choosing the right tool for the job!

std::stable\_sort (gcc) - 8950 comparisons, 20268 array accesses, 1.00 ms delay

<http://panthema.net/2013/sound-of-sorting>



What's next?



# Roadmap

## Object-Oriented Programming

### C++ basics

User/client

**vectors + grids**

**stacks + queues**

**sets + maps**

Implementation

arrays

dynamic memory management

linked data structures

real-world algorithms

*Life after CS106B!*

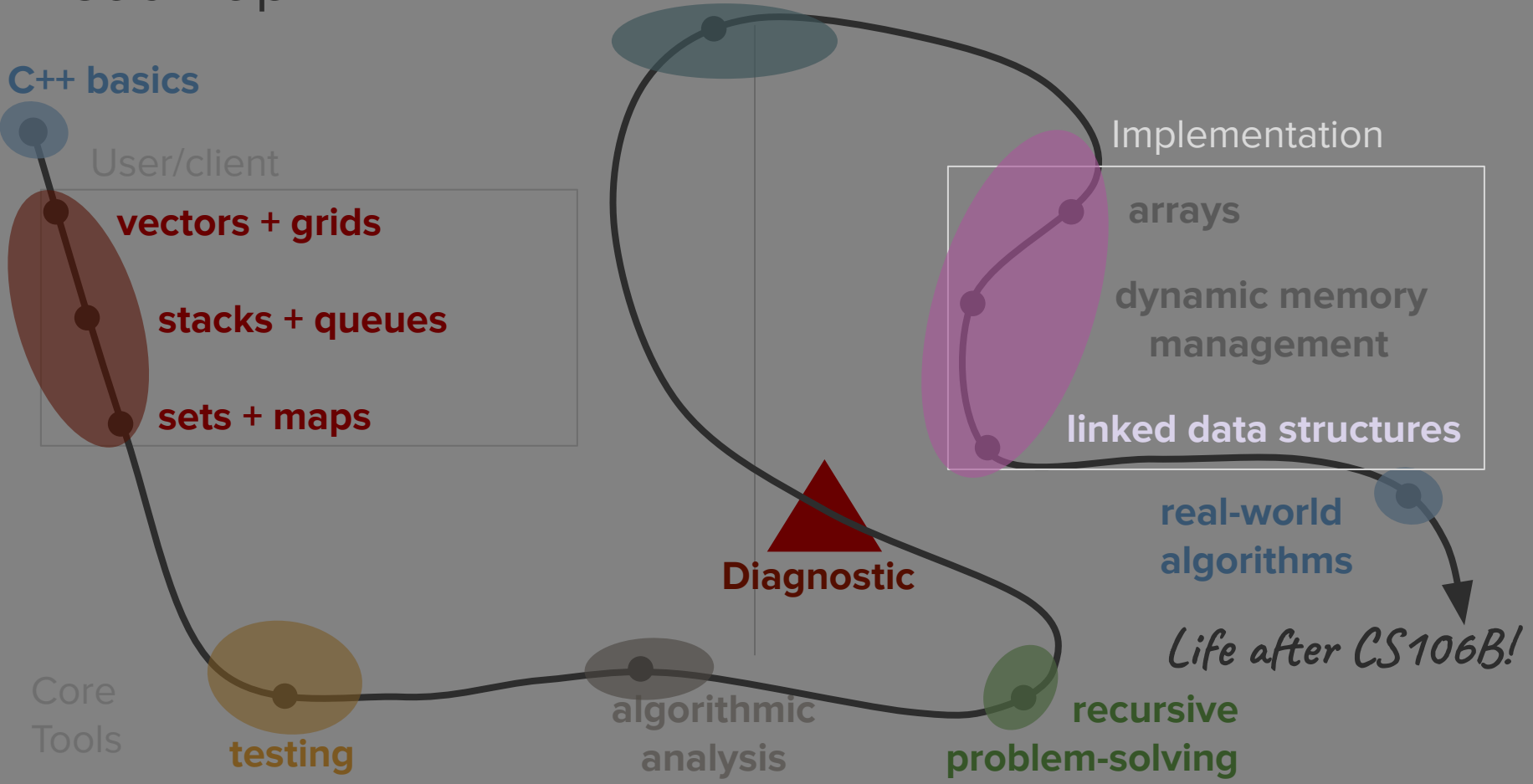
Core Tools

**testing**

algorithmic analysis

**recursive problem-solving**

**Diagnostic**



Trees!

